Asian Journal of Advanced Research and Reports

Volume 18, Issue 7, Page 256-265, 2024; Article no.AJARR.118368
ISSN: 2582-3248

Quasi-Rational Solutions to the Seventh Equation of the NLS Hierarchy

Pierre Gaillard [a*]

[a] *Université de Bourgogne Franche Comté, Institut de Mathématiques de Bourgogne,9 avenue Alain Savary BP 47870, 21078 Dijon Cedex, France.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: https://doi.org/10.9734/ajarr/2024/v18i7700

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/118368

Received: 14/04/2024
Accepted: 19/06/2024
Published: 29/06/2024

Review Article

ABSTRACT

The following study is part of a research program of rational solutions of the hierarchy of the nonlinear Schrödinger equation. Here, we are interested in the equation of order 7 and we highlight particular solutions providing the first orders of rogue waves not yet found.

Keywords: NLS hierarchy; quasi-rational solutions.

PACS Numbers: 33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd.

1 INTRODUCTION

Quasi-rational solutions to the seventh equation of the NLS hierarchy are constructed. We give explicit expressions of these solutions for the first orders. They depend on multi-parameters and so patterns of these solutions in the (x, t) plane according the different values of the parameters are studied.

**Corresponding author: E-mail: Pierre.Gaillard@u-bourgogne.fr, pgaillar@u-bourgogne.fr;*

Cite as: *Gaillard, Pierre. 2024. "Quasi-Rational Solutions to the Seventh Equation of the NLS Hierarchy". Asian Journal of Advanced Research and Reports 18 (7):256-65. https://doi.org/10.9734/ajarr/2024/v18i7700.*

We consider the seventh equation of the NLS hierarchy of order 7 $(NLS7)$ which can be written as

$$\begin{array}{l} iu_t + u_{8x} + 16|u^2|u_{6x} + 2u^2\overline{u}_{6x} + 56\overline{u}u_x u_{5x} \\ +40u\overline{u}_x u_{5x} + 12uu_x\overline{u}_{5x} + 98|u|^4 u_{4x} + 168|u_x|^2 u_{4x} \\ +112\overline{u}u_{2x}u_{4x} + 72u\overline{u}_{2x}u_{4x} + 28u^2|u|^2\overline{u}_{4x} + 42u_x^2\overline{u}_{4x} \\ +44uu_{2x}\overline{u}_{4x} + 68uu_x\overline{u}_{3x} + 476|u|^2\overline{u}u_x u_{3x} + 252u_x\overline{u}_{2x}u_{3x} \\ +308u|u|^2\overline{u}_x u_{3x} + 308\overline{u}_x u_{2x}u_{3x} + 70\overline{u}u_{3x}^2 + 196u_x u_{2x}\overline{u}_{3x} \\ +168u|u|^2 u_x\overline{u}_{3x} + 56u^3\overline{u}_x\overline{u}_{3x} + 280|u|^6 u_{2x} + 1456|u|^2|u_x|^2 u_{2x} \\ +490\overline{u}^2 u_x^2 u_{2x} + 238u^2\overline{u}_x^2 u_{2x} + 588|u|^2 u_x^2\overline{u}_{2x} + 336u^2|u_x|^2\overline{u}_{2x} \\ +140|u|^4 u^2\overline{u}_{2x} + 42u^3(\overline{u}_{2x})^2 + 392|u|^2 u|u_{2x}|^2 + 322|u|^2\overline{u}u_{2x}^2 \\ +182u_{2x}^2\overline{u}_{2x} + 560|u|^4\overline{u}u_x^2 + 560|u|^4 u|u_x|^2 + 420\overline{u}u_x^2|u_x|^2 \\ +140u^3|u|^2\overline{u}_x^2 + 378|u_x|^4 u + 70|u|^8 u \end{array} \tag{1}$$

with as usual the subscripts meaning partial derivatives and $\overline{u}$ the complex conjugate of u.

Different classical equations are included in the NLS hierarchy; the first one is the NLS equation [1, 2, 3, 4, 5, 6, 7, 8, 9]; the second one is the mKdV equation [10, 11, 12, 13, 14]; the third one is the LPD equation [15, 16, 17, 18, 19].

Many works has been done for these first three equations of the NLS hierarchy. For example, we can quote the following works, for the NLS equation [20], the mKdV equation [11], the LPD equation [15, 21, 22]. However, very few studies have been carried out for the following orders of this hierarchy. Here we explicitly construct solutions of the order equation seven of this hierarchy.

We construct quasi rational solutions for the first orders. The related patterns of the modulus of these solutions in the plane of coordinates $(x;t)$ are studied.

2 QUASI RATIONAL SOLUTIONS OF ORDER 1 TO THE NLS7 EQUATION

Theorem 2.1. *The function $v(x,t)$ defined by*

$$v(x,t) = -\frac{\left(3 - 4\,x^2 - 313600\,t^2 + 2240\,it\right) \mathrm{e}^{70\,it}}{1 + 4\,x^2 + 313600\,t^2} \tag{2}$$

is a solution to the $(NLS7)$ equation (1).

Proof: We have to replace the expression of the solution given by (2) and check that (1) is verified.

We get a smooth solution of the equation (1).

3 QUASI RATIONAL SOLUTIONS OF ORDER 2 OF THE NLS7 EQUATION DEPENDING ON 2 REAL PARAMETERS

Theorem 3.1. *The function $v(x,t)$ defined by*

$$v(x,t) = \frac{n(x,t)}{d(x,t)} \tag{3}$$

with

$n(x,t) = -(-64\,x^6 + 2304\,b_1 x^5 - 768\,{a_1}^2 x^4 + 144\,x^4 + 107520\,itx^4 - 215040\,a_1 tx^4 + 768\,ia_1 x^4 - 15052800\,t^2 x^4$

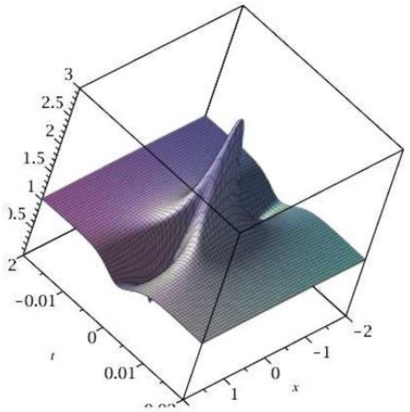

Fig. 1. Solution of order 1 to $(NLS7)$.

$$- 34560\, {b_1}^2x^4 - 18432\, ia_1x^3b_1 - 4992\, b_1x^3 + 276480\, {b_1}^3x^3 + 18432\, b_1{a_1}^2x^3 + 361267200\, b_1t^2x^3 + 5160960\, b_1a_1tx^3 - 2580480\, itx^3b_1 + 2580480\, i{a_1}^2tx^2 - 1244160\, {b_1}^4x^2 + 2903040\, a_1tx^2 - 3072\, {a_1}^4x^2 + 361267200\, ia_1t^2x^2 - 33718272000\, a_1t^3x^2 + 23224320\, it{b_1}^2x^2 - 165888\, {b_1}^2{a_1}^2x^2 + 293529600\, t^2x^2 - 1180139520000\, t^4x^2 + 5760\, {a_1}^2x^2 + 16859136000\, it^3x^2 - 1152\, ia_1x^2 + 58752\, {b_1}^2x^2 + 6144\, i{a_1}^3x^2 - 361267200\, {a_1}^2t^2x^2 + 165888\, ia_1{b_1}^2x^2 - 1720320\, {a_1}^3tx^2 - 3251404800\, {b_1}^2t^2x^2 - 46448640\, {b_1}^2a_1tx^2 - 806400\, itx^2 + 180\, x^2 - 202309632000\, it^3xb_1 - 3161088000\, b_1t^2x + 36864\, b_1{a_1}^4x - 29675520\, b_1a_1tx + 663552\, {b_1}^3{a_1}^2x - 50688\, b_1{a_1}^2x - 5616\, b_1x + 20643840\, b_1{a_1}^3tx - 663552\, ia_1x{b_1}^3 - 290304\, {b_1}^3x + 13005619200\, {b_1}^3t^2x - 4608\, ia_1xb_1 + 4335206400\, b_1{a_1}^2t^2x + 14161674240000\, b_1t^4x + 7096320\, itb_1x - 4335206400\, ia_1t^2xb_1 + 185794560\, {b_1}^3a_1tx - 92897280\, itx{b_1}^3 - 73728\, i{a_1}^3xb_1 - 30965760\, i{a_1}^2txb_1 + 2985984\, {b_1}^5x + 404619264000\, b_1a_1t^3x - 262080\, it + 23602790400000\, ia_1t^4 + 54792192000\, it^3 - 1475174400000\, t^4 + 3225600\, i{a_1}^2t + 13005619200\, ia_1t^2{b_1}^2 + 337182720000\, i{a_1}^2t^3 + 606928896000\, it^3{b_1}^2 + 73543680\, {b_1}^2a_1t - 30840979456000000\, t^6 + 139345920\, it{b_1}^4 + 8601600\, i{a_1}^4t + 660878131200000\, it^5 + 69120\, ia_1{b_1}^2 - 13547520\, it{b_1}^2 - 13005619200\, {b_1}^2{a_1}^2t^2 - 1213857792000\, {b_1}^2a_1t^3 - 278691840\, {b_1}^4a_1t + 995328\, ia_1{b_1}^4 + 221184\, i{a_1}^3{b_1}^2 - 45 + 812851200\, ia_1t^2 - 98784000\, t^2 + 92897280\, i{a_1}^2t{b_1}^2 + 3010560\, {a_1}^3t + 270950400\, {a_1}^2t^2 - 8429568000\, a_1t^3 - 61931520\, {b_1}^2{a_1}^3t + 96768\, {b_1}^2{a_1}^2 + 8399462400\, {b_1}^2t^2 + 1872\, {a_1}^2 + 18000\, {b_1}^2 + 846720\, a_1t - 720\, ia_1 + 1536\, i{a_1}^3 + 12288\, i{a_1}^5 + 2408448000\, i{a_1}^3t^2 - 42485022720000\, {b_1}^2t^4 - 995328\, {b_1}^4{a_1}^2 - 19508428800\, {b_1}^4t^2 - 110592\, {b_1}^2{a_1}^4 - 23602790400000\, {a_1}^2t^4 - 3440640\, {a_1}^5t - 1204224000\, {a_1}^4t^2 - 224788480000\, {a_1}^3t^3 - 1321756262400000\, a_1t^5 - 2985984\, {b_1}^6 - 4096\, {a_1}^6 + 518400\, {b_1}^4 + 8448\, {a_1}^4)\mathrm{e}^{2\, i(a_1+35\, t)}$$

and

$$d(x,t) = 64\, x^6 - 2304\, b_1x^5 + 48\, x^4 + 15052800\, t^2x^4 + 215040\, a_1tx^4 + 768\, {a_1}^2x^4 + 34560\, {b_1}^2x^4 - 5160960\, b_1a_1tx^3 - 361267200\, b_1t^2x^3 - 276480\, {b_1}^3x^3 + 384\, b_1x^3 - 18432\, b_1{a_1}^2x^3 + 1180139520000\, t^4x^2 + 361267200\, {a_1}^2t^2x^2 + 46448640\, {b_1}^2a_1tx^2 - 203212800\, t^2x^2 + 1720320\, {a_1}^3tx^2 - 1152\, {a_1}^2x^2 - 1612800\, a_1tx^2 + 1244160\, {b_1}^4x^2 + 3072\, {a_1}^4x^2 + 165888\, {b_1}^2{a_1}^2x^2 + 3251404800\, {b_1}^2t^2x^2 - 17280\, {b_1}^2x^2 + 108\, x^2 + 33718272000\, a_1t^3x^2 - 20643840\, b_1{a_1}^3tx - 185794560\, {b_1}^3a_1tx - 4335206400\, b_1{a_1}^2t^2x - 404619264000\, b_1a_1t^3x - 2985984\, {b_1}^5x - 13005619200\, {b_1}^3t^2x + 2077286400\, b_1t^2x + 124416\, {b_1}^3x - 14161674240000\, b_1t^4x - 663552\, {b_1}^3{a_1}^2x - 4608\, b_1{a_1}^2x - 2448\, b_1x + 14192640\, b_1a_1tx - 36864\, b_1{a_1}^4x + 9 + 177020928000\, a_1t^3 + 19508428800\, {b_1}^4t^2 + 347155200\, t^2 - 5148057600\, {b_1}^2t^2 + 7375872000000\, t^4 + 30840979456000000\, t^6 + 69120\, {b_1}^2{a_1}^2 + 1411200\, a_1t + 110592\, {b_1}^2{a_1}^4 + 995328\, {b_1}^4{a_1}^2 - 27095040\, {b_1}^2a_1t + 23602790400000\, {a_1}^2t^4 + 3440640\, {a_1}^5t + 1204224000\, {a_1}^4t^2 + 224788480000\, {a_1}^3t^3 + 1321756262400000\, a_1t^5 + 2985984\, {b_1}^6 + 4096\, {a_1}^6 - 269568\, {b_1}^4 + 6912\, {a_1}^4 + 42485022720000\, {b_1}^2t^4 + 5591040\, {a_1}^3t + 1535385600\, {a_1}^2t^2 + 61931520\, {b_1}^2{a_1}^3t + 13005619200\, {b_1}^2{a_1}^2t^2 + 1213857792000\, {b_1}^2a_1t^3 + 278691840\, {b_1}^4a_1t + 1584\, {a_1}^2 + 20016\, {b_1}^2$$

is a solution to the (NLS7) equation (1).

Proof: We have also to replace the expression of the solution given by (3), and we check that the relation (1) is verified.

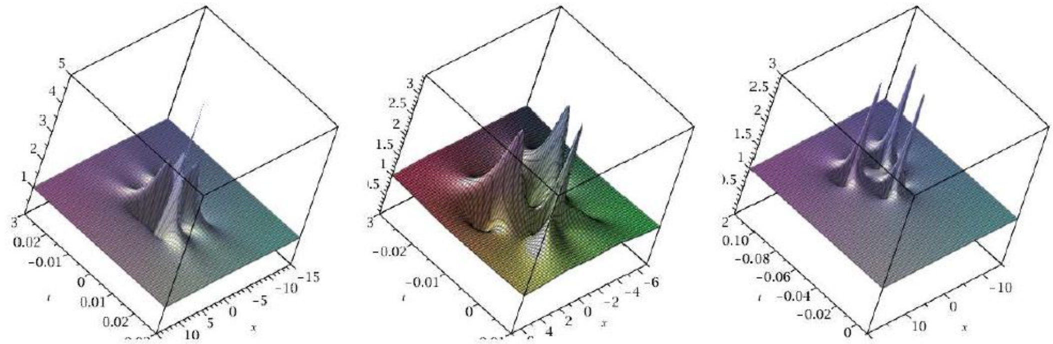

Fig. 2. Solution of order 2 to the equation (1); to the left $a_1 = 0$, $b_1 = 0$; in the center $a_1 = 1$, $b_1 =$; to the right $a_1 = 10$, $b_1 = 0$.

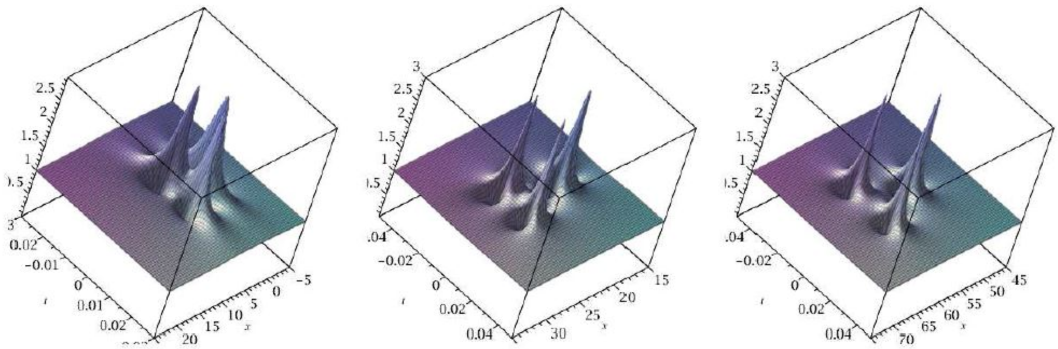

Fig. 3. Solution of order 2 to the equation (1); to the left $a_1 = 0$, $b_1 = 1$; in the center $a_1 = 0$, $b_1 = 4$; to the right $a_1 = 0$, $b_1 = 10$.

When one or both parameters increase, three peaks appear. When only one of the parameters increases, the three peaks appear but with different orientations.

4 QUASI RATIONAL SOLUTIONS OF ORDER 3 OF THE NLS7 EQUATION

The solution of order 3, depending on 4 real parameters being too long, we only present in the appendix. Here we give the solution of order 3 without parameters.

Theorem 4.1. *The function* $v(x,t)$ *defined by*

$$v(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4}$$

with

$$n(x,t) = -(-4096\,x^{12}+13762560\,itx^{10}-1926758400\,t^2x^{10}+18432\,x^{10}-377644646400000\,t^4x^8+93929472000\,t^2x^8$$
$$-\,258048000\,ix^8t + 57600\,x^8 + 5394923520000\,it^3x^8 - 167242629120000\,ix^6t^3 + 20770455552000000\,t^4x^6$$
$$-\,1369202688000\,t^2x^6 + 172800\,x^6 + 1109606400\,itx^6 + 845924007936000000\,it^5x^6 - 39476453703680000000\,t^6x^6$$
$$-\,1712996116070400000\,ix^4t^5 + 716083200\,itx^4 + 875169792000\,t^2x^4 - 952878366720000\,ix^4t^3$$
$$+\,663204422221824000000000\,it^7x^4 + 150997435416576000000\,t^6x^4 - 226800\,x^4 + 59903882035200000\,t^4x^4$$
$$-\,2321215477776384000000000\,t^8x^4 - 653632074547200000\,t^4x^2 + 313364089499811840000000000\,t^8x^2$$
$$+\,267893559263232000000\,it^5x^2 - 7585324185600\,t^2x^2 - 727933173830674022400000000000\,t^{10}x^2$$
$$-\,2024697964894617600000\,t^6x^2 - 99480663333273600000000\,ix^2t^7 + 2599761335109550080000000000\,it^9x^2$$
$$+\,3826686689280000\,it^3x^2 + 3969907200\,itx^2 - 113400\,x^2 + 407642577345177452544000000000000\,it^{11}$$
$$+\,433641600\,it - 5496461881344000\,it^3 + 24845295667485081600000000\,it^7 - 264443775418368000000\,it^5$$
$$+\,5546744179200\,t^2 + 1679109246236491776000000000\,t^8 + 26647553684872888320000000000\,it^9 + 14175$$
$$-\,30937159887803645952000000000\,t^{10} - 951166013805414055936000000000000\,t^{12} - 704818560983040000\,t^4$$
$$+\,5505521933824819200000\,t^6)\mathrm{e}^{70\,it}$$

and

$$d(x,t) = 4096\,x^{12} + 1926758400\,t^2x^{10} + 6144\,x^{10} + 34560\,x^8 - 65028096000\,t^2x^8 + 377644646400000\,t^4x^8$$
$$+\,877879296000\,t^2x^6 - 13217562624000000\,t^4x^6 + 149760\,x^6 + 39476453703680000000\,t^6x^6$$
$$+\,2321215477776384000000000\,t^8x^4 - 1562032668672000000\,t^4x^4 - 680968826388480000000\,t^6x^4$$
$$-\,3248695296000\,t^2x^4 + 54000\,x^4 + 48600\,x^2 + 2149048794061209600000\,t^6x^2 + 1886547433881600000\,t^4x^2$$
$$+\,2236763289600\,t^2x^2 + 727933173830674022400000000000\,t^{10}x^2 + 10445469649993728000000000\,t^8x^2$$
$$-\,13357721658064896000000\,t^6 + 1110098090091777884160000000000\,t^{10}$$
$$+\,281244270326081126400000000\,t^8 + 866898922659840000\,t^4$$
$$+\,951166013805414055936000000000000\,t^{12} + 2025 + 3320852774400\,t^2$$

is a solution to the (NLS) *equation* (1).

Proof: It is sufficient to check that the relation (1) is verified when we replace the expression of the solution given by (5).

In the following, we give the patterns of the modules of the solutions according to different values of the parameters.

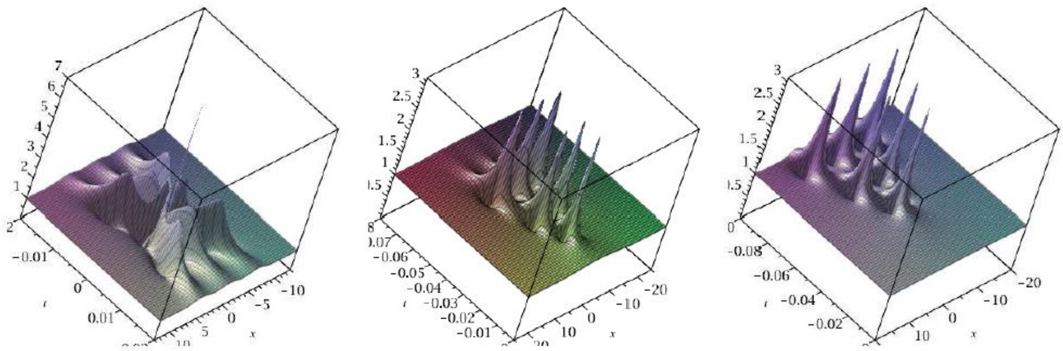

Fig. 4. Solution of order 3 **to** (1)**; to the left** $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$**; in the center** $a_1 = 1$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$**; to the right** $a_1 = 10$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$**.**

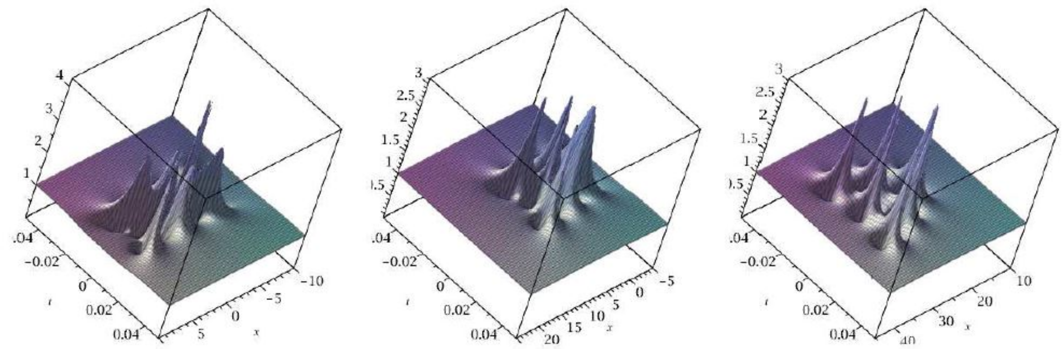

Fig. 5. Solution of order 3 to (1); to the left $a_1 = 0$, $b_1 = 0, 1$, $a_2 = 0$, $b_2 = 0$; in the center $a_1 = 0$, $b_1 = 5$, $a_2 = 0$, $b_2 = 0$; to the right $a_1 = 0$, $b_1 = 5$, $a_2 = 0$, $b_2 = 0$.

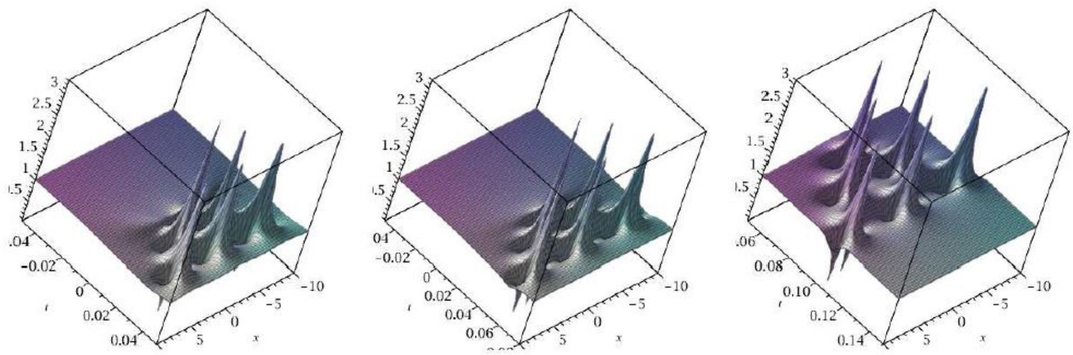

Fig. 6. Solution of order 3 to (1); to the left $a_1 = 0$, $b_1 = 0$, $a_2 = 0, 5$, $b_2 = 0$; in the center $a_1 = 0$, $b_1 = 0$, $a_2 = 1$, $b_2 = 0$; to the right $a_1 = 0$, $b_1 = 5$, $a_2 = 2$, $b_2 = 0$.

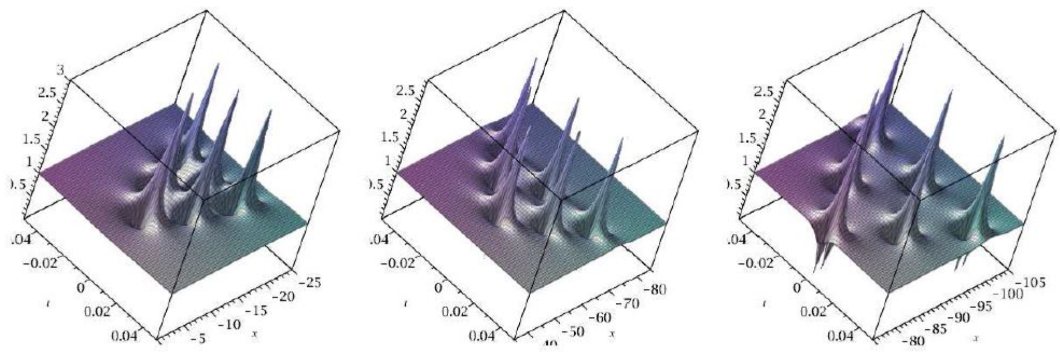

Fig. 7. Solution of order 3 to (1); to the left $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0, 5$; in the center $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 2$; to the right $a_1 = 0$, $b_1 = 5$, $a_2 = 0$, $b_2 = 3$.

We remark the similarity with these solutions and those relative to other equations belonging to this NLS hierarchy. For example, we recover the same types of patterns like in the NLS equation [23], the mKdV equation [24], or the Lakshmanan Porsezian Daniel equation [25]. We get the structure of triangles with peaks which appear in function of the different values of the parameters.

5 CONCLUSION

Quasi-rational solutions to the $(NLS7)$ equation have been constructed for the first orders. These N-order solutions appear as the quotient of a polynomial of degree $N(N+1)$ in x and t for the numerator by a polynomial of degree $N(N+1)$ in x and t for the denominator.

The solutions of order 2 depend on two real parameters, and the structure of triangles with three peaks is observed for their modules.

The solutions of order 3 depend on four real parameters. In the plane (x,t) of the coordinates, the representation of the modules of the solutions reveals the formation of triangles containing 6 peaks.

It will be relevant to study other solutions of this equations and study the patterns of their modulus.

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

[1] Zakharov VE. Stability of periodic waves of finite amplitude on a surface of a deep fluid. J. Appl. Tech. Phys. 1968;9:86-94.

[2] Akhmediev N, Ankiewicz A, Soto-Crespo JM. Rogues waves and rational solutions of nonlinear Schrödinger equation. Phys. Rev. E, V. 2009;80:026601-1-9.

[3] Akhmediev N, Ankiewicz A. First-order exact solutions of the nonlinear Schrödinger equation in the normal-dispersion regime. Phys. Rev. A, V. 2009;47(4):3213-3221.

[4] Ankiewicz A, Kedziora DJ, Akhmediev N. Rogue wave triplets. Phys Lett. A, V. 2011;375:2782-2785.

[5] Dubard P, Gaillard P, Klein C, Matveev VB. On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation. Eur. Phys. J. Spe. Top. 2010;185:247-258.

[6] Eleonskii V, Krichever I, Kulagin N. Rational multisoliton solutions of nonlinear Schrödinger equation. Dokl. Math. Phy. 1986;287:606-610.

[7] Gaillard P. Other 2N-2 parameters solutions to the NLS equation and 2N+1 highest amplitude of the modulus of the N-th order AP breather. J. Phys. A: Math. Theor. 2015;48:145203-1-23.

[8] Gaillard P. Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves. J. Math. Phys. 2013;54;013504-1-32.

[9] Gaillard P. Families of quasi-rational solutions of the NLS equation and multi-rogue waves. J. Phys. A : Meth. Theor., 2011;44:1-15.

[10] Tanaka S. Modified korteweg. de Vries equation and scattering theory. Proc. Japan Acad. 1972;48:466-469.

[11] Wadati M. The exact solution of the modified Korteweg-de Vries equation. J. Phys. Soc. Jpn. 1972;32:1681-1681.

[12] Ono Y. Algebraic soliton of the modified Korteweg-de Vries equation. Jour. Phys. Soc.Japan, 1976;41(5):1817-1818.

[13] Chowdury A, Ankiewicz A, Akhmediev N. Periodic and rational solutions of modified Korteweg-de Vries equation. Eur. Phys. J. D, V. 2016;70(104):1-7.

[14] Chowdury A, Ankiewicz A, Akhmediev N. Periodic and rational solutions of mKdV equation. Eur. Phys. J. D, V. 2016;70(104):1-7.

[15] Lakshmanan M, Porsezian K, Daniel M. Effect of discreteness on the continuum limit of the Heisenberg spin chain. Phys. Lett. A. 1998;133(9):483-488.

[16] Akram G, Sadaf M, Dawood M, Baleanu D. Optical solitons for Lakshmanan Porsezian Daniel equation with Kerr law non-linearity using improved tan expansion technique. Res. In Phys. 2021;29:104758-1-13.

[17] Al Qarni AA, et al. Optical solitons for lakshmanan porsezian daniel model by riccati equation approach. Optik. 2019;182:922-929.

[18] Alqahtani RT, Babatin MM, Biswas A. Bright optical solitons for Lakshmanan Porsezian Daniel model by semi-inverse variational principle. Optik, V. 2018;154:109-114.

[19] Arshed S, et al. Optical solitons in birefringent fibers for Lakshmanan Porsezian Daniel model using exp(-iϕ)-expansion method. Optik. 2018;172:651-656.

[20] Akhmediev N, Eleonski V, Kulagin N. Generation of periodic trains of picosecond pulses in an optical fiber : Exact solutions. Sov. Phys. J.E.T.P. 1985;62:894-899.

[21] Lakshmanan M, Daniel M, Porsezian K. On the integrability aspects of the one-dimensional classical continuum isotropic biquadratic Heisenberg spin chain. J. Math. Phys. 1992;33(5):1807-1816.

[22] Daniel M, Porsezian K, Lakshmanan M. On the integrable models of the higher order water wave equation. Phys. Lett. A. 1993;174(3):237-240.

[23] Gaillard P. Towards a classification of the quasi rational solutions to the NLS equation. Theor. And Math. Phys. 2016;189(1):1440-1449.

[24] Gaillard P. Rational solutions to the mKdV equation associated to particular polynomials. Wave Motion. 2021;107:102824-1-11.

[25] Gaillard P. Rogue waves of the lakshmanan porsezian daniel equation depending on multi-parameters. As. Jour. Of Adv. Res. And Rep. 2022;16(3):32-40.

APPENDIX

Solution of order 3 to the (NLS5) equation depending on 4 real parameters :

The function $v(x,t)$ defined by

$$v(x,t) = \left(1 - 24\frac{n(x,t)}{d(x,t)}\right) \mathrm{e}^{i(2\,a_1-6\,a_2+70\,t)} \tag{5}$$

with

$$n(x,t) = (675 + 69363302400\,t^2 + 88473600\,{b_2}^2 + (2\,x - 12\,b_1 + 60\,b_2)^{10} + 27000\,(8\,b_1 - 80\,b_2)^2 + 91800\,(16\,a_2 -
1120\,t)^2 + 2190\,(4\,a_1 - 24\,a_2 + 560\,t)^6 + 495\,(4\,a_1 - 24\,a_2 + 560\,t)^8 + 11\,(4\,a_1 - 24\,a_2 + 560\,t)^{10} - 720\,(4\,a_1 -
24\,a_2 + 560\,t)^7(16\,a_2 - 1120\,t) - 3600\,(4\,a_1 - 24\,a_2 + 560\,t)^3(48\,a_2 - 15904\,t) - 720\,(4\,a_1 - 24\,a_2 + 560\,t)^5(272\,a_2 -
25312\,t) - 154828800\,(16\,a_2 - 1120\,t)t + i(15422400\,t + 64800\,(16\,a_2 - 1120\,t)^3 - 870\,(4\,a_1 - 24\,a_2 + 560\,t)^7 + 25\,(4\,a_1 -
24\,a_2 + 560\,t)^9 + (4\,a_1 - 24\,a_2 + 560\,t)^{11} - 151200\,a_2 - 5529600\,(8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t)b_2 - 77414400\,(16\,a_2 -
1120\,t)^2t - 90\,(4\,a_1 - 24\,a_2 + 560\,t)^8(16\,a_2 - 1120\,t) - 120\,(4\,a_1 - 24\,a_2 + 560\,t)^6(80\,a_2 - 11872\,t) + 900\,(4\,a_1 - 24\,a_2 +
560\,t)^4(464\,a_2 - 23520\,t) + (-450\,(4\,a_1 - 24\,a_2 + 560\,t)^3 - 210\,(4\,a_1 - 24\,a_2 + 560\,t)^5 + 10\,(4\,a_1 - 24\,a_2 + 560\,t)^7 +
300\,(4\,a_1 - 24\,a_2 + 560\,t)^4(16\,a_2 - 1120\,t) + 450\,(4\,a_1 - 24\,a_2 + 560\,t)(-3 + 12\,(8\,b_1 - 80\,b_2)^2 - 4\,(16\,a_2 - 1120\,t)^2) -
14400\,a_2 + 2620800\,t + 1800\,(4\,a_1 - 24\,a_2 + 560\,t)^2(16\,a_2 - 2016\,t))(2\,x - 12\,b_1 + 60\,b_2)^4 + (-480\,(4\,a_1 - 24\,a_2 +
560\,t)^5(8\,b_1 - 80\,b_2) + 14400\,(4\,a_1 - 24\,a_2 + 560\,t)^2(8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t) + 7200\,(4\,a_1 - 24\,a_2 + 560\,t)(8\,b_1 -
48\,b_2) - 2400\,(4\,a_1 - 24\,a_2 + 560\,t)^3(16\,b_1 - 128\,b_2) - 14400\,(8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t) - 460800\,(16\,a_2 - 1120\,t)b_2 +
12902400\,(8\,b_1 - 80\,b_2)t)(2\,x - 12\,b_1 + 60\,b_2)^3 - 21600\,(8\,b_1 - 80\,b_2)^2(16\,a_2 - 1120\,t) + (1710\,(4\,a_1 - 24\,a_2 + 560\,t)^5 -
60\,(4\,a_1 - 24\,a_2 + 560\,t)^7 + 5\,(4\,a_1 - 24\,a_2 + 560\,t)^9 - 900\,(4\,a_1 - 24\,a_2 + 560\,t)^3(7 + 4\,(8\,b_1 - 80\,b_2)^2 - 12\,(16\,a_2 -
1120\,t)^2) + 675\,(4\,a_1 - 24\,a_2 + 560\,t)(7 + 16\,(8\,b_1 - 80\,b_2)^2 + 16\,(16\,a_2 - 1120\,t)^2) - 345600\,a_2 + 38707200\,t - 21600\,(8\,b_1 -
80\,b_2)^2(16\,a_2 - 1120\,t) - 21600\,(16\,a_2 - 1120\,t)^3 + 9676800\,(4\,a_1 - 24\,a_2 + 560\,t)^2t - 1800\,(4\,a_1 - 24\,a_2 + 560\,t)^4(64\,a_2 -
1792\,t))(2\,x - 12\,b_1 + 60\,b_2)^2 + (-240\,(4\,a_1 - 24\,a_2 + 560\,t)^7(8\,b_1 - 80\,b_2) - 7200\,(4\,a_1 - 24\,a_2 + 560\,t)^4(8\,b_1 -
80\,b_2)(16\,a_2 - 1120\,t) + 10800\,(4\,a_1 - 24\,a_2 + 560\,t)(24\,b_1 - 400\,b_2 + 4\,(8\,b_1 - 80\,b_2)^3 + 4\,(8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t)^2) +
3600\,(4\,a_1 - 24\,a_2 + 560\,t)^3(24\,b_1 - 176\,b_2) + 720\,(4\,a_1 - 24\,a_2 + 560\,t)^5(56\,b_1 - 400\,b_2) + 21600\,(8\,b_1 - 80\,b_2)(16\,a_2 -
1120\,t) + 1382400\,(16\,a_2 - 1120\,t)b_2 - 38707200\,(8\,b_1 - 80\,b_2)t - 43200\,(4\,a_1 - 24\,a_2 + 560\,t)^2((8\,b_1 - 80\,b_2)(16\,a_2 -
1120\,t) + 32\,(16\,a_2 - 1120\,t)b_2 - 896\,(8\,b_1 - 80\,b_2)t))(2\,x - 12\,b_1 + 60\,b_2) + 90\,(4\,a_1 - 24\,a_2 + 560\,t)^5(-107 + 28\,(8\,b_1 -
80\,b_2)^2 + 12\,(16\,a_2 - 1120\,t)^2) + 5400\,(4\,a_1 - 24\,a_2 + 560\,t)^2(176\,a_2 - 22176\,t + 4\,(8\,b_1 - 80\,b_2)^2(16\,a_2 - 1120\,t) +
4\,(16\,a_2 - 1120\,t)^3) - 225\,(4\,a_1 - 24\,a_2 + 560\,t)^3(11 + 80\,(8\,b_1 - 80\,b_2)^2 + 80\,(16\,a_2 - 1120\,t)^2 + 4096\,(8\,b_1 - 80\,b_2)b_2 +
114688\,(16\,a_2 - 1120\,t)t) - 675\,(4\,a_1 - 24\,a_2 + 560\,t)(-7 + 56\,(8\,b_1 - 80\,b_2)^2 + 88\,(16\,a_2 - 1120\,t)^2 - 4096\,(8\,b_1 - 80\,b_2)b_2 -
131072\,{b_2}^2 - 102760448\,t^2) + 77414400\,(8\,b_1 - 80\,b_2)^2t + (4\,a_1 - 24\,a_2 + 560\,t)(2\,x - 12\,b_1 + 60\,b_2)^{10} + (-60\,a_1 +
840\,a_2 - 42000\,t + 5\,(4\,a_1 - 24\,a_2 + 560\,t)^3)(2\,x - 12\,b_1 + 60\,b_2)^8 + (-600\,a_1 - 240\,a_2 + 722400\,t - 140\,(4\,a_1 - 24\,a_2 +
560\,t)^3 + 10\,(4\,a_1 - 24\,a_2 + 560\,t)^5 + 240\,(4\,a_1 - 24\,a_2 + 560\,t)^2(16\,a_2 - 1120\,t))(2\,x - 12\,b_1 + 60\,b_2)^6 + (-240\,(4\,a_1 -
24\,a_2 + 560\,t)^3(8\,b_1 - 80\,b_2) - 1440\,(8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t) + 720\,(4\,a_1 - 24\,a_2 + 560\,t)(8\,b_1 - 176\,b_2))(2\,x - 12\,b_1 +
60\,b_2)^5) + 15\,(1 + (4\,a_1 - 24\,a_2 + 560\,t)^2)(2\,x - 12\,b_1 + 60\,b_2)^8 + (210 - 60\,(4\,a_1 - 24\,a_2 + 560\,t)^2 + 50\,(4\,a_1 - 24\,a_2 +
560\,t)^4 + 480\,(4\,a_1 - 24\,a_2 + 560\,t)(16\,a_2 - 1120\,t))(2\,x - 12\,b_1 + 60\,b_2)^6 + (-720\,(4\,a_1 - 24\,a_2 + 560\,t)^2(8\,b_1 - 80\,b_2) -
5760\,b_1 - 11520\,b_2)(2\,x - 12\,b_1 + 60\,b_2)^5 + (450\,(4\,a_1 - 24\,a_2 + 560\,t)^2 - 150\,(4\,a_1 - 24\,a_2 + 560\,t)^4 + 70\,(4\,a_1 - 24\,a_2 +
560\,t)^6 + 1200\,(4\,a_1 - 24\,a_2 + 560\,t)^3(16\,a_2 - 1120\,t) - 450 + 5400\,(8\,b_1 - 80\,b_2)^2 - 1800\,(16\,a_2 - 1120\,t)^2 + 3600\,(4\,a_1 -
24\,a_2 + 560\,t)(16\,a_2 - 2016\,t))(2\,x - 12\,b_1 + 60\,b_2)^4 + (-2400\,(4\,a_1 - 24\,a_2 + 560\,t)^4(8\,b_1 - 80\,b_2) + 28800\,(4\,a_1 - 24\,a_2 +
560\,t)(8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t) + 57600\,b_1 - 806400\,b_2 - 7200\,(4\,a_1 - 24\,a_2 + 560\,t)^2(16\,b_1 - 128\,b_2))(2\,x - 12\,b_1 +
60\,b_2)^3 + (6750\,(4\,a_1 - 24\,a_2 + 560\,t)^4 + 420\,(4\,a_1 - 24\,a_2 + 560\,t)^6 + 45\,(4\,a_1 - 24\,a_2 + 560\,t)^8 - 2700\,(4\,a_1 - 24\,a_2 +
560\,t)^2(5 + 4\,(8\,b_1 - 80\,b_2)^2 - 12\,(16\,a_2 - 1120\,t)^2) - 675 - 10800\,(8\,b_1 - 80\,b_2)^2 - 10800\,(16\,a_2 - 1120\,t)^2 + 21600\,(4\,a_1 -
24\,a_2 + 560\,t)(32\,a_2 - 3136\,t) - 7200\,(4\,a_1 - 24\,a_2 + 560\,t)^3(32\,a_2 + 448\,t))(2\,x - 12\,b_1 + 60\,b_2)^2 + (-1680\,(4\,a_1 - 24\,a_2 +
560\,t)^6(8\,b_1 - 80\,b_2) - 28800\,(4\,a_1 - 24\,a_2 + 560\,t)^3(8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t) - 10800\,(4\,a_1 - 24\,a_2 + 560\,t)^2(8\,b_1 -
272\,b_2) + 86400\,b_1 - 1209600\,b_2 + 43200\,(8\,b_1 - 80\,b_2)^3 + 43200\,(8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t)^2 + 3600\,(4\,a_1 - 24\,a_2 +
560\,t)^4(8\,b_1 + 80\,b_2) - 86400\,(4\,a_1 - 24\,a_2 + 560\,t)((8\,b_1 - 80\,b_2)(16\,a_2 - 1120\,t) + 32\,(16\,a_2 - 1120\,t)b_2 - 896\,(8\,b_1 -
80\,b_2)t))(2\,x - 12\,b_1 + 60\,b_2) + 450\,(4\,a_1 - 24\,a_2 + 560\,t)^4(-17 + 28\,(8\,b_1 - 80\,b_2)^2 + 12\,(16\,a_2 - 1120\,t)^2) + 10800\,(4\,a_1 -
24\,a_2 + 560\,t)(-16\,a_2 + 2016\,t + 4\,(8\,b_1 - 80\,b_2)^2(16\,a_2 - 1120\,t) + 4\,(16\,a_2 - 1120\,t)^3) + 675\,(4\,a_1 - 24\,a_2 + 560\,t)^2(-3 +
16\,(8\,b_1 - 80\,b_2)^2 + 16\,(16\,a_2 - 1120\,t)^2 - 4096\,(8\,b_1 - 80\,b_2)b_2 - 114688\,(16\,a_2 - 1120\,t)t) - 2764800\,(8\,b_1 - 80\,b_2)b_2)$$

and

$d(x,t) = 2024+416179814400\,t^2+530841600\,{b_2}^2+356400\,(8\,b_1-80\,b_2)^2+518400\,(8\,b_1-80\,b_2)^4+874800\,(16\,a_2-1120\,t)^2+3720\,(4\,a_1-24\,a_2+560\,t)^8+120\,(4\,a_1-24\,a_2+560\,t)^{10}+518400\,(16\,a_2-1120\,t)^4+(1+(2\,x-12\,b_1+60\,b_2)^2+(4\,a_1-24\,a_2+560\,t)^2)^6+(-360\,(4\,a_1-24\,a_2+560\,t)^8(8\,b_1-80\,b_2)-17280\,(4\,a_1-24\,a_2+560\,t)^5(8\,b_1-80\,b_2)(16\,a_2-1120\,t)-1440\,(4\,a_1-24\,a_2+560\,t)^6(8\,b_1-240\,b_2)+32400\,(4\,a_1-24\,a_2+560\,t)^4(8\,b_1+112\,b_2)+64800\,(4\,a_1-24\,a_2+560\,t)^2(-40\,b_1+752\,b_2+4\,(8\,b_1-80\,b_2)^3+4\,(8\,b_1-80\,b_2)(16\,a_2-1120\,t)^2)-777600\,(4\,a_1-24\,a_2+560\,t)((8\,b_1-80\,b_2)(16\,a_2-1120\,t)+64\,(16\,a_2-1120\,t)b_2-1792\,(8\,b_1-80\,b_2)t)-172800\,(4\,a_1-24\,a_2+560\,t)^3(3\,(8\,b_1-80\,b_2)(16\,a_2-1120\,t)+32\,(16\,a_2-1120\,t)b_2-896\,(8\,b_1-80\,b_2)t)-648000\,b_1+8553600\,b_2+259200\,(8\,b_1-80\,b_2)^3+1296000\,(8\,b_1-80\,b_2)(16\,a_2-1120\,t)^2-33177600\,(8\,b_1-80\,b_2)^2b_2+33177600\,(16\,a_2-1120\,t)^2b_2-1857945600\,(8\,b_1-80\,b_2)(16\,a_2-1120\,t)t)(2\,x-12\,b_1+60\,b_2)+80\,(4\,a_1-24\,a_2+560\,t)^6(191+63\,(8\,b_1-80\,b_2)^2+27\,(16\,a_2-1120\,t)^2)+21600\,(4\,a_1-24\,a_2+560\,t)^3(-368\,a_2+23072\,t+4\,(8\,b_1-80\,b_2)^2(16\,a_2-1120\,t)+4\,(16\,a_2-1120\,t)^3)+120\,(8\,b_1-80\,b_2)(2\,x-12\,b_1+60\,b_2)^9+46080\,b_2(2\,x-12\,b_1+60\,b_2)^7-1161216000\,(16\,a_2-1120\,t)t+240\,(4\,a_1-24\,a_2+560\,t)^4(599+135\,(8\,b_1-80\,b_2)^2-225\,(16\,a_2-1120\,t)^2-11520\,(8\,b_1-80\,b_2)b_2-322560\,(16\,a_2-1120\,t)t)-16200\,(4\,a_1-24\,a_2+560\,t)(496\,a_2-52640\,t+80\,(8\,b_1-80\,b_2)^2(16\,a_2-1120\,t)+16\,(16\,a_2-1120\,t)^3+4096\,(8\,b_1-80\,b_2)(16\,a_2-1120\,t)b_2-57344\,(8\,b_1-80\,b_2)^2t+57344\,(16\,a_2-1120\,t)^2t)+24\,(4\,a_1-24\,a_2+560\,t)^2(3881+12150\,(8\,b_1-80\,b_2)^2+28350\,(16\,a_2-1120\,t)^2+691200\,(8\,b_1-80\,b_2)b_2+22118400\,{b_2}^2+17340825600\,t^2)+(-120\,(4\,a_1-24\,a_2+560\,t)^2+360\,(4\,a_1-24\,a_2+560\,t)(16\,a_2-1120\,t)+120)(2\,x-12\,b_1+60\,b_2)^8+(480\,(4\,a_1-24\,a_2+560\,t)^2-240\,(4\,a_1-24\,a_2+560\,t)^4+960\,(4\,a_1-24\,a_2+560\,t)^3(16\,a_2-1120\,t)+2320+2160\,(8\,b_1-80\,b_2)^2+5040\,(16\,a_2-1120\,t)^2-1440\,(4\,a_1-24\,a_2+560\,t)(64\,a_2-8960\,t))(2\,x-12\,b_1+60\,b_2)^6+(-720\,(4\,a_1-24\,a_2+560\,t)^4(8\,b_1-80\,b_2)-17280\,(4\,a_1-24\,a_2+560\,t)(8\,b_1-80\,b_2)(16\,a_2-1120\,t)+4320\,(4\,a_1-24\,a_2+560\,t)^2(8\,b_1-176\,b_2)-51840\,b_1+103680\,b_2)(2\,x-12\,b_1+60\,b_2)^5+(-1440\,(4\,a_1-24\,a_2+560\,t)^4+720\,(4\,a_1-24\,a_2+560\,t)^5(16\,a_2-1120\,t)+240\,(4\,a_1-24\,a_2+560\,t)^2(56+135\,(8\,b_1-80\,b_2)^2-45\,(16\,a_2-1120\,t)^2)+32400\,(4\,a_1-24\,a_2+560\,t)(16\,a_2-2912\,t)+7200\,(4\,a_1-24\,a_2+560\,t)^3(48\,a_2-4256\,t)+3360+32400\,(8\,b_1-80\,b_2)^2-54000\,(16\,a_2-1120\,t)^2+2764800\,(8\,b_1-80\,b_2)b_2+77414400\,(16\,a_2-1120\,t)t)(2\,x-12\,b_1+60\,b_2)^4+(-960\,(4\,a_1-24\,a_2+560\,t)^6(8\,b_1-80\,b_2)+57600\,(4\,a_1-24\,a_2+560\,t)^3(8\,b_1-80\,b_2)(16\,a_2-1120\,t)-43200\,(4\,a_1-24\,a_2+560\,t)^2(24\,b_1-272\,b_2)-7200\,(4\,a_1-24\,a_2+560\,t)^4(48\,b_1-448\,b_2)+345600\,b_1-5529600\,b_2-86400\,(8\,b_1-80\,b_2)^3-86400\,(8\,b_1-80\,b_2)(16\,a_2-1120\,t)^2+172800\,(4\,a_1-24\,a_2+560\,t)((8\,b_1-80\,b_2)(16\,a_2-1120\,t)-32\,(16\,a_2-1120\,t)b_2+896\,(8\,b_1-80\,b_2)t))(2\,x-12\,b_1+60\,b_2)^3+(13440\,(4\,a_1-24\,a_2+560\,t)^6+240\,(4\,a_1-24\,a_2+560\,t)^8-240\,(4\,a_1-24\,a_2+560\,t)^4(-326+45\,(8\,b_1-80\,b_2)^2-135\,(16\,a_2-1120\,t)^2)+480\,(4\,a_1-24\,a_2+560\,t)^2(-76+135\,(8\,b_1-80\,b_2)^2+1215\,(16\,a_2-1120\,t)^2)-129600\,(4\,a_1-24\,a_2+560\,t)^3(32\,a_2-1344\,t)-12960\,(4\,a_1-24\,a_2+560\,t)^5(32\,a_2-1344\,t)-64800\,(4\,a_1-24\,a_2+560\,t)(-96\,a_2+11200\,t+4\,(8\,b_1-80\,b_2)^2(16\,a_2-1120\,t)+4\,(16\,a_2-1120\,t)^3)+12144-97200\,(8\,b_1-80\,b_2)^2+32400\,(16\,a_2-1120\,t)^2+530841600\,{b_2}^2-464486400\,(16\,a_2-1120\,t)t+416179814400\,t^2)(2\,x-12\,b_1+60\,b_2)^2-2160\,(4\,a_1-24\,a_2+560\,t)^5(240\,a_2-47264\,t)-1440\,(4\,a_1-24\,a_2+560\,t)^7(80\,a_2-6496\,t)-120\,(4\,a_1-24\,a_2+560\,t)^9(16\,a_2-1120\,t)+1036800\,(8\,b_1-80\,b_2)^2(16\,a_2-1120\,t)^2-24883200\,(8\,b_1-80\,b_2)b_2$

is a solution to the $(NLS5)$ equation (1).

Peer-review history:
The peer review history for this paper can be accessed here:
https://www.sdiarticle5.com/review-history/118368