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Research Article

Solving Nonlinear Fractional Partial Differential Equations Using the Elzaki Transform Method and the Homotopy Perturbation Method

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In this paper, we combine the Elzaki transform method (ETM) with the new homotopy perturbation method (NHPM) for the first time. This hybrid approach can solve initial value problems numerically and analytically, such as nonlinear fractional differential equations of various normal orders. The Elzaki transform method (ETM) is used to solve nonlinear fractional differential equations, and then the homotopy is applied to the transformed equation, which includes the beginning conditions. To obtain the solution to an equation, we use the inverse transforms of the Elzaki transform method (ETM). The initial conditions have a big impact on the equation's result. We give three beginning value issues that were solved as precise or approximation solutions with high rigor to demonstrate the method's power and correctness. It is clear that solving nonlinear partial differential equations with the crossbred approach is the best alternative.

1. Introduction

Many scholars have been investigating and debating the linear and nonlinear fractional differential equations (FDEs) in recent years. In view of the many applications of fractional differential equations in assorted fields, which engender in the physical sciences as well as in engineering, these kinds of equations play a worthy turn and also help to evolve mathematical tools to realize fractional modeling.

The Elzaki transform is a transform similar to integral by other metamorphoses defined by integrals, which are known as Laplace transforms and Sumudu transform. In solving linear and non-linear differential equations. Using these method help in whereas the conversion was known by Tarig M. Elzaki [1]. Admit for its performance in solving linear order, nonlinear partial differential equations, and integral equations, the interesting convert it is evidence in [2–4]. The homotopy analysis transform method (HARM) is one of the more technicalities utilize in the solutions for the nonlinear factor [5]. To solve it, the homotopy perturbation technique combines the Laplace transform method and the well-known base flow equation [6]. By placing the solution in a rapid approximation series, HPM paired with the Sumudu transform tool improves the answer in a closed shape [7]. The theoretical formulation of initial value issues for fractional differential equations may be done in two methods [8]. The homotopy perturbation method (HPM) was introduced by Ji-Huan He in 1999 [9] for solving differential and integral equations. The HPM is applied to algebraic equations [10], nonlinear reaction-diffusion-convection problem [11], singular boundary and initial value problems [12, 13], nonlinear wave equations [14], and other modifications which can be seen in [15–25].

The major goal of this research is to combine the Elzaki transform (ET) with the new homotopy perturbation

Table 1: Solution for the first three approximations with exact solution, with mesh points $\delta = 0.2$, for equation (18).

δ	μ	$\nu = 0.75$	$\nu = 0.85$	$\nu = 0.95$	$\nu = 1$	Exact
	0	0	0	0	0	0
	0.1	-0.1480375	-0.136454922	-0.128120827	-0.1248	-0.125
	0.2	-0.296074999	-0.272909844	-0.256241654	-0.2496	-0.25
	0.3	-0.444112499	-0.409364766	-0.384362481	-0.3744	-0.375
	0.4	-0.592149999	-0.545819688	-0.512483308	-0.4992	-0.5
0.2	0.5	-0.740187498	-0.68227461	-0.640604134	-0.624	-0.625
	0.6	-0.888224998	-0.818729532	-0.768724961	-0.7488	-0.75
	0.7	-1.036262498	-0.955184454	-0.896845788	-0.8736	-0.875
	0.8	-1.184299998	-1.091639376	-1.024966615	-0.9984	-1
	0.9	-1.332337497	-1.228094299	-1.153087442	-1.1232	-1.125
	1	-1.480374997	-1.364549221	-1.281208269	-1.248	-1.25

method (NHPM) to approximate the solution of specific initial value problems represented by highly nonlinear fractional partial differential equations with starting conditions. Our suggested technique, which is a combination of the Elzaki transform (ET) and the new homotopy perturbation method (NHPM), discovers precise or approximate solutions to initial value problems with a high rate of convergence of the solution series.

The methodology of this paper includes the following: item 1 contains definitions of fractional derivatives and discussion of an advanced method on the solution. Item 2 contains an explanation of the method to a solution. Item 3 uses the method to resolve some of the FPDEs to illustrate the approximate accuracy.

Definition 1. Let$u(\delta)$, for $\delta > 0$,be in the spaceC_θ, $\theta \in \mathbb{R}$,if there exist a real number$v > \theta$,per se$u(\delta) = \delta^v u_1(\delta)$, where$u_1(\delta) \in C[0,\infty)$, visibly$C_\theta \subset C_v$, if $v \leq \theta$,[9 , 26 , 27].

Definition 2. Let taking the operator of order $\upsilon > 0$, Riemann-Liouville fractional integral to $u(\delta) \in C_\eta$, $\eta \geq -1$, is known as [27]

$$\begin{aligned} J^\upsilon h(\tau) &= \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - \xi)^{\upsilon - 1} h(\xi) d\xi \qquad \upsilon, \tau, \xi > 0, \\ J^\upsilon u(\delta) &= \frac{1}{\Gamma(\nu)} \int_0^\delta (\delta - \xi)^{\upsilon - 1} u(\xi) d(\xi) \quad \upsilon, \delta, \xi > 0, \end{aligned} \tag{1}$$

Definition 3. Caputo fractional derivative, the left party of $j \in C_{-1}^w$, $w \in \mathrm{N} \cup \{0\}$, is known as [26]

$$D^\upsilon h(\tau) = \frac{\partial^\upsilon h(\tau)}{\partial \tau^\upsilon} = J^{\vartheta - \nu} \left[\frac{\partial^\vartheta h(\tau)}{\partial \tau^\vartheta} \right], \quad \vartheta - 1 < \upsilon \leq \vartheta, \vartheta \in N. \tag{2}$$

Definition 4. If the set [1]

$$A = \left\{ |u(\delta)| < N\, e^{\delta/\kappa_i} \text{ if } \delta \in (-1)^i \times [0,\infty) \right\}. \tag{3}$$

Then Elzaki integral transform (EIT) of a function $u(\delta)$ is defined as

$$E[u(\delta)] = \beta \int_0^\infty u(\delta)\, e^{-\delta/\beta} d\delta = U(\beta) \quad \kappa_1 \leq \beta \leq \kappa_2. \tag{4}$$

Definition 5. If $\vartheta - 1 < \upsilon \leq \vartheta$, $\vartheta \in \mathrm{N}$, then (EIT) of the fractional derivative $D_*^\nu u(\mu, \delta)$ is

$$\mathrm{E}[D_*^\nu u(\mu, \delta)]] = \frac{U(\mu, \beta)}{\beta^\nu} - \sum_{\kappa=0}^{r-1} \beta^{2-\nu+\kappa} u^{(\kappa)}(\mu, 0), \vartheta - 1 < \upsilon \leq \vartheta, \tag{5}$$

where $U(\mu, \beta)$ be the (EIT) $u(\mu, \delta)$ [28].

2. Elzaki Transform Homotopy Perturbation Method (ETHPM)

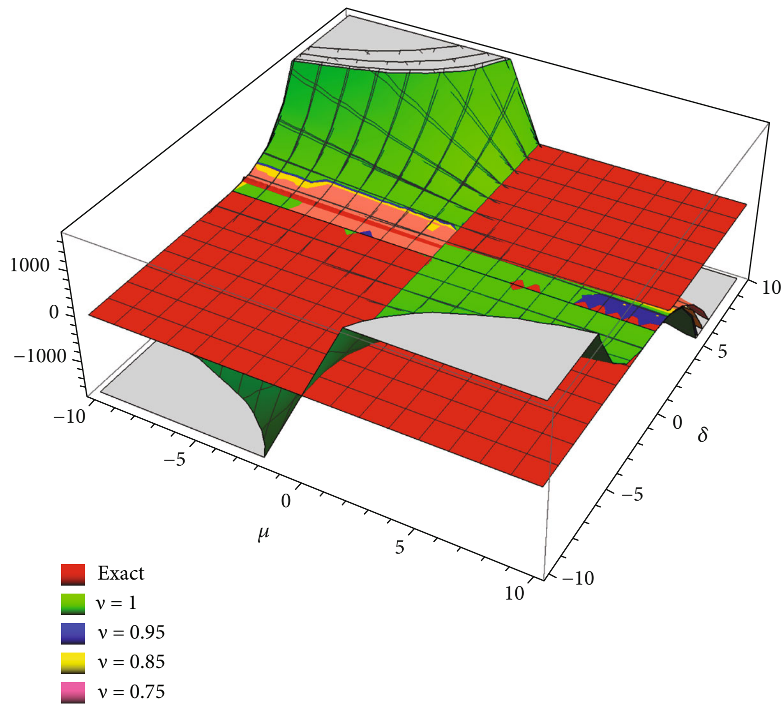
This section intends to discuss the utilization of (ETHPM) algorithm to solve linear and nonlinear fraction partial differential equations.

$$D_\delta^\nu u(\mu, \delta) + Ru(\mu, \delta) + \mathrm{N}u(\mu, \delta) = \Phi(\mu, \delta), \mu, \delta \geq 0, r - 1 < \nu \leq r, \tag{6}$$

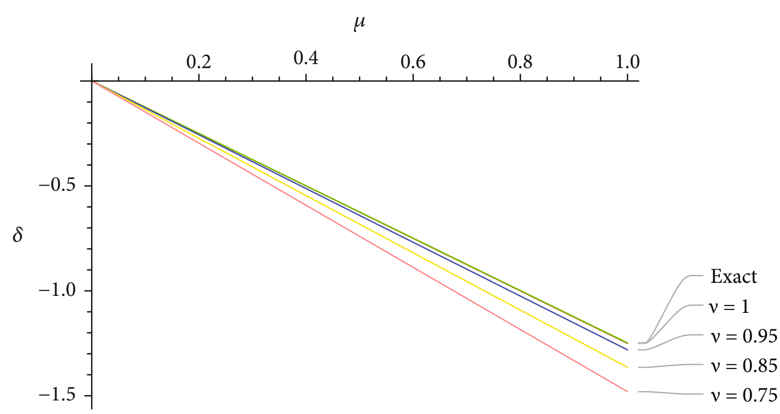
where $D_\delta^\nu = \partial^\nu / \partial \delta^\nu$ represents the order ν fractional derivative, R represents a linear operator, N represents a nonlinear function, and Φ represents the source function. The beginning and boundary conditions are determined using Equation (6).

$$\begin{aligned} u(\mu, 0) &= \varphi(\mu), \quad 0 < \nu \leq 1, \\ u(\mu, 0) &= \phi(\mu), \\ \frac{\partial u(\mu, 0)}{\partial \delta} &= \varphi(\mu), \end{aligned} \tag{7}$$

$$1 < \nu \leq 2. \tag{8}$$



(a) 3D approximate solution Example 1



(b) 2D approximate solution Example 1

FIGURE 1: Create 2D and 3D comparison charts of precise data and predicted computational outputs for Example 1, using $\delta = 0.2$.

Using the linearity of (ET) and applying it to both aspect equations (6), the conclusion is

$$\mathrm{E}[D_\delta^\nu u(\mu,\delta)] + \mathrm{E}[Ru(\mu,\delta)] + \mathrm{E}[Nu(\mu,\delta)] = \mathrm{E}[\Phi(\mu,\delta)], \quad \nu > 0. \tag{9}$$

Using the property of (ET), to get

$$\frac{u(\mu,\delta)}{\beta^\nu} - C + \mathrm{E}[Ru(\mu,\delta)] + \mathrm{E}[Nu(\mu,\delta)] = \mathrm{E}[\Phi(\mu,\delta)], \quad \nu > 0. \tag{10}$$

where $C = \sum_{\kappa=0}^{n-1} \beta^{2-\nu+\kappa} u^{(\kappa)}(\mu, 0)$,

$$u(\mu,\beta) = \beta^\nu \mathrm{E}[\Phi(\mu,\delta)] + \beta^\nu C - \beta^\nu \mathrm{E}[Ru(\mu,\delta)] - \beta^\nu \mathrm{E}[Nu(\mu,\delta)]. \tag{11}$$

Taking the inverse (ET) to both aspect equations (11), we get

$$u(\mu,\delta) = G(\mu,\delta) - \mathrm{E}^{-1}[\beta^\nu \mathrm{E}[Ru(\mu,\delta) + Nu(\mu,\delta)]]. \tag{12}$$

This is now applied to the HPM, where $G(\mu,\delta)$ is the term from the start conditions.

$$u(\mu,\delta) = \sum_{\kappa=0}^{\infty} P^\kappa u_\kappa(\mu,\delta). \tag{13}$$

Table 2: Solution for the first three approximations with exact solution, with mesh points $\delta = 0.2$, for equation (26).

δ	μ	$\nu = 0.75$	$\nu = 0.85$	$\nu = 0.95$	$\nu = 1$	Exact
	0	1.403184045	1.315919781	1.248964885	1.221333333	1.221402758
	0.1	1.550758199	1.454316273	1.380319669	1.349782081	1.349858808
	0.2	1.713852863	1.607268051	1.525489155	1.491739902	1.491824698
	0.3	1.894100342	1.776305907	1.68592625	1.648627557	1.648721271
	0.4	2.093304614	1.96312163	1.863236662	1.822015231	1.8221188
0.2	0.5	2.313459382	2.169584934	2.059194972	2.013638245	2.013752707
	0.6	2.556768029	2.397762174	2.275762398	2.225414428	2.225540928
	0.7	2.82566567	2.649937023	2.515106419	2.459463307	2.459603111
	0.8	3.122843523	2.928633332	2.77962247	2.718127321	2.718281828
	0.9	3.451275843	3.236640389	3.071957917	3.003995266	3.004166024
	1	3.814249692	3.57704083	3.395038551	3.319928206	3.320116923

The nonlinear operator is decomposition as, accordingly:

$$Nu(\mu, \delta) = \sum_{\kappa=0}^{\infty} P^{\kappa} H_n(u), \tag{14}$$

where $H_n(u)$ are given by

$$H_n(u_0, u_1, \cdots u_n) = \frac{1}{n!}\frac{\partial}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 1, 2, \cdots. \tag{15}$$

When we make up the equation (14) and equation (13) into equation (12), we get

$$\sum_{n=0}^{\infty} P^n u_n(\mu, \delta) = G(\mu, \delta) - P\mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[R\sum_{n=0}^{\infty} u_n(\mu, \delta) + N\sum_{n=0}^{\infty} P^n H(u_n)\right]\right]. \tag{16}$$

Using He's polynomials, this is the conjugation of the ET and the HPM. We find after matching the coefficient

$$\begin{aligned}
P^0 &: u_0(\mu, \delta) = G(\mu, \delta),\\
P^1 &: u_1(\mu, \delta) = -\mathrm{E}^{-1}[\beta^{\nu}\mathrm{E}[Ru_0(\mu, \delta) + H_0(u)]],\\
P^2 &: u_2(\mu, \delta) = -\mathrm{E}^{-1}[\beta^{\nu}\mathrm{E}[Ru_1(\mu, \delta) + H_1(u)]],\\
P^3 &: u_3(\mu, \delta) = -\mathrm{E}^{-1}[\beta^{\nu}\mathrm{E}[Ru_2(\mu, \delta) + H_2(u)]],\\
&\vdots \quad \vdots \quad \vdots
\end{aligned} \tag{17}$$

Then the solution is $u(\mu, \delta) = \lim_{p \longrightarrow 1} u_n(\mu, \delta) = u_0(\mu, \delta) + u_1(\mu, \delta) + u_2(\mu, \delta) + \cdots$.

3. Illustrative Application

Here, we append examples to explain the solve method described.

Example 1. Let the homogeneous nonlinear fractional partial differential equations [29].

$$D_{\delta}^{\nu} u + uu_{\mu} = 0, 0 < \nu \leq 1. \tag{18}$$

with initial condition:

$$u(\mu, 0) = -\mu. \tag{19}$$

Taking the (ET) to both aspect equations (18) and subject to equation (19), we get

$$u(\mu, \beta) = -\mu\beta^2 - \beta^{\nu}\mathrm{E}[u(\mu, \delta)u_{\delta}(\mu, \delta)]. \tag{20}$$

Taking the inverse of (ET), we get:

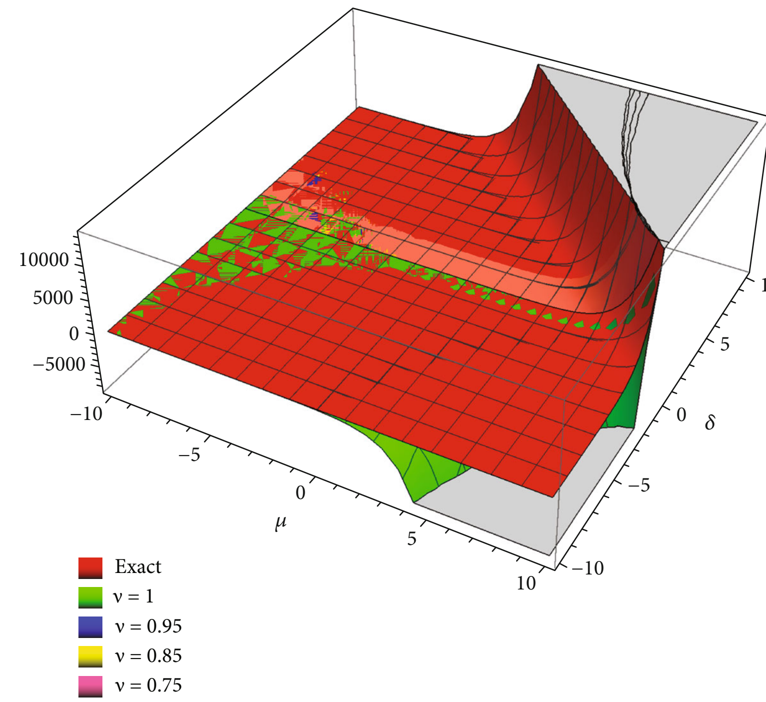
$$u(\mu, \delta) = -\mu - \mathrm{E}^{-1}\left[\beta^{\alpha}\mathrm{E}\left[u(\mu, \delta)u_{\mu}(\mu, \delta)\right]\right]. \tag{21}$$

When, we apply the (HPM), we get:

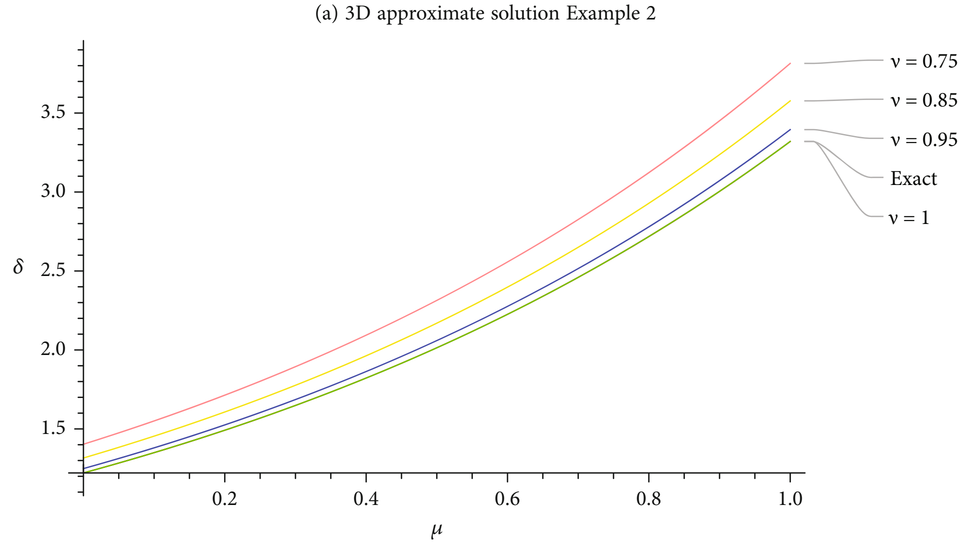
$$\sum_{\kappa=0}^{\infty} P^{\kappa} u_{\kappa}(\mu, \delta) = -\mu - \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[\sum_{\kappa=0}^{\infty} P^{\kappa} H_{\kappa}(u)\right]\right]. \tag{22}$$

After matching the coefficient we find:

$$\begin{aligned}
P^0 &: u_0(\mu, \delta) = -\mu,\\
P^1 &: u_1(\mu, \delta) = -\mathrm{E}^{-1}[\beta^{\nu}\mathrm{E}[H_0(u)]] = -\mu\frac{\delta^{\nu}}{\nu!},\\
P^2 &: u_2(\mu, \delta) = -\mathrm{E}^{-1}[\beta^{\nu}\mathrm{E}[H_1(u)]] = -2\mu\frac{\delta^{2\nu}}{(2\nu)!},\\
p^3 &: u_3(\mu, \delta) = -\mathrm{E}^{-1}[\beta^{\alpha}\mathrm{E}[H_2(u)]]\\
&= -\mu\left[\frac{4}{(2\nu)!} + \frac{1}{(\nu)^2!}\right]\frac{(2\nu)!\delta^{3\alpha}}{(3\nu)!},\\
&\vdots \quad \vdots \quad \vdots
\end{aligned} \tag{23}$$



(a) 3D approximate solution Example 2



(b) 2D approximate solution Example 2

FIGURE 2: Create 2D and 3D comparison charts of precise data and predicted computational outputs for Example 2, using $\delta = 0.2$.

The convergent solution of equation (18), is presented by

$$u(\mu,\delta) = -\mu\left[1 + \frac{\delta^{\nu}}{\nu!} + 2\frac{\delta^{2\nu}}{2\nu!} + \left[\frac{4}{(2\nu)!} + \frac{1}{(\nu)^2!}\right]\frac{(2\nu)!\delta^{3\nu}}{(3\nu)!} + \cdots\right]. \tag{24}$$

The convergent solution of equation (18) at $\nu \longrightarrow 1$

$$u(\mu,\delta) = -\mu\left(1 + \delta + \delta^2 + \delta^3 + \cdots\right) = \frac{\mu}{\delta - 1}, \tag{25}$$

we have the exact solution [29], see Table 1 and Figure 1.

Example 2. Let a homogeneous nonlinear diffusion problem [30].

$$D_{\delta}^{\nu}u = u_{\mu\mu} - u_{\mu} + uu_{\mu\mu} - u^2 + u, 0 < \nu \leq 1 \tag{26}$$

have initial condition:

$$u(\mu, 0) = e^{\mu}. \tag{27}$$

Taking the (ET) to both aspect equations (26) and subject to equation (27), we get

$$u(\mu,\beta) = \beta^2 e^{\mu} + \beta^{\nu}\mathrm{E}\left[\left(u_{\mu\mu} - u_{\mu} + u\right) + \left(uu_{\mu\mu} - u^2\right)\right]. \tag{28}$$

Table 3: Solution for the first three approximations with exact solution, with mesh points $\delta = 0.2$, for equation (34).

δ	μ	$\nu = 1.75$	$\nu = 1.85$	$\nu = 1.95$	$\nu = 2$	Exact
	0	0	0	0	0	0
	0.1	0.020005412	0.020004087	0.020003077	0.020002667	0.020002667
	0.2	0.040043345	0.04003272	0.04002463	0.040021347	0.040021347
	0.3	0.060146586	0.060110585	0.060083207	0.060072104	0.060072104
	0.4	0.080348451	0.080262642	0.080197497	0.080171104	0.080171105
0.2	0.5	0.100683062	0.100514271	0.100386409	0.10033467	0.100334672
	0.6	0.121185631	0.120891415	0.120669137	0.120579331	0.120579337
	0.7	0.141892761	0.141420729	0.141065245	0.140921878	0.140921895
	0.8	0.162842761	0.162129738	0.161594737	0.161379417	0.161379461
	0.9	0.184075989	0.183047001	0.182278142	0.181969427	0.181969529
	1	0.20563521	0.204202283	0.203136596	0.202709821	0.202710036

Taking the inverse of (ET), we get

$$u(\mu,\delta) = e^{\mu} + \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[\left(u_{\mu\mu} - u_{\mu} + u\right) - \left(uu_{\mu\mu} - u^2\right)\right]\right]. \tag{29}$$

When, we apply the (HPM), we get

$$\sum_{\kappa=0}^{\infty} P^n u_n(\mu,\delta) = e^{\mu} + \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[\sum_{\kappa=0}^{\infty} P^n u_n(\mu,\delta) + \sum_{\kappa=0}^{\infty} P^n H_n(u)\right]\right]. \tag{30}$$

The elementary few components of He's polynomials [5, 6] are offered by

$$\begin{aligned} H_0(u) &= u_0(u_0)_{\mu\mu} - u_0^2, \\ H_1(u) &= u_0(u_1)_{\mu\mu} + u_1(u_0)_{\mu\mu} - 2u_0u_1, \\ H_2(u) &= u_0(u_2)_{\mu\mu} + u_1(u_1)_{\mu\mu} + u_2(u_0)_{\mu\mu} - u_1^2 - 2u_2u_0, \\ &\vdots \end{aligned} \tag{31}$$

After matching the coefficient we find

$$\begin{aligned} P^0 &: u_0(\mu,\delta) = e^{\mu}, \\ P^1 &: u_1(\mu,\delta) = \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[u_{0\mu\mu} - u_{0\mu} + u_0 + H_0(u)\right]\right] = \frac{\delta^{\nu}}{(\nu)!}e^{\mu}, \\ P^2 &: u_2(\mu,\delta) = \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[u_{1\mu\mu} - u_{1\mu} + u_1 + H_1(u)\right]\right] = \frac{\delta^{2\nu}}{(2\nu)!}e^{\mu}, \\ P^3 &: u_3(\mu,\delta) = \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[u_{2\mu\mu} - u_{2\mu} + u_2 + H_2(u)\right]\right] = \frac{\delta^{3\nu}}{(3\nu)!}e^{\mu}, \\ &\vdots \quad \vdots \quad \vdots \end{aligned} \tag{32}$$

The convergent solution of equation (26), is presented by

$$u(\mu,\delta) = e^{\mu}\left[1 + \frac{\delta^{\nu}}{(\nu)!} + \frac{\delta^{2\nu}}{(2\nu)!} + \frac{\delta^{3\nu}}{(3\nu)!} + \cdots\right], = e^{\mu}\sum_{n=1}^{\infty}\frac{\delta^{n\nu}}{(n\nu)!}. \tag{33}$$

If we take $\nu \longrightarrow 1$, we get exact solution of $u(\mu,\delta) = e^{\mu+\delta}$ [30]; see Table 2 and Figure 2.

Example 3. Let a homogeneous nonlinear fractional partial differential equations [31].

$$D_{\delta}^{\nu}u - 2\frac{\mu^2}{\delta}uu_{\mu} = 0, \delta \geq 0, 1 < \nu \leq 2. \tag{34}$$

with initial condition

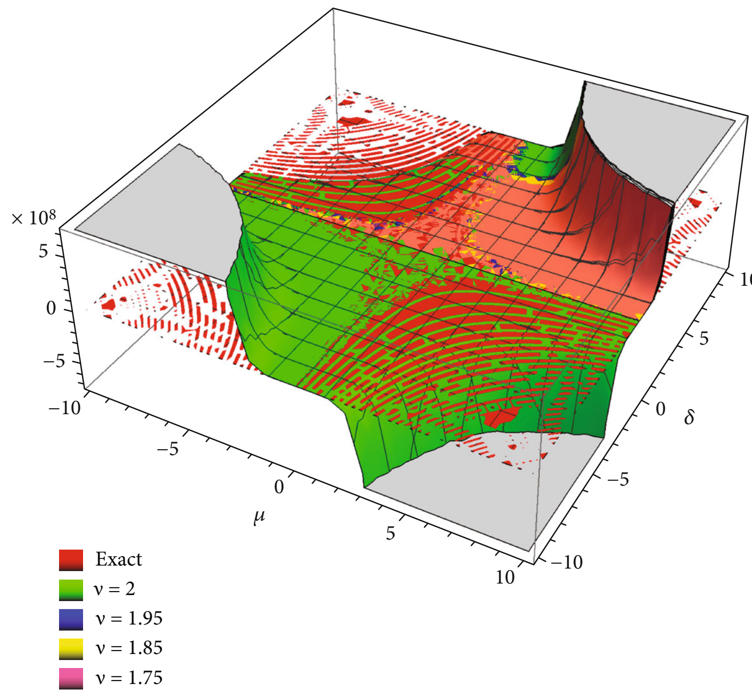
$$\vdots \qquad \begin{aligned} u(\mu,0) &= 0, \\ u_{\mu}(\mu,0) &= \mu. \end{aligned} \tag{35}$$

Taking the (ET) to both aspect equations (34) and subject to equation (35), we get

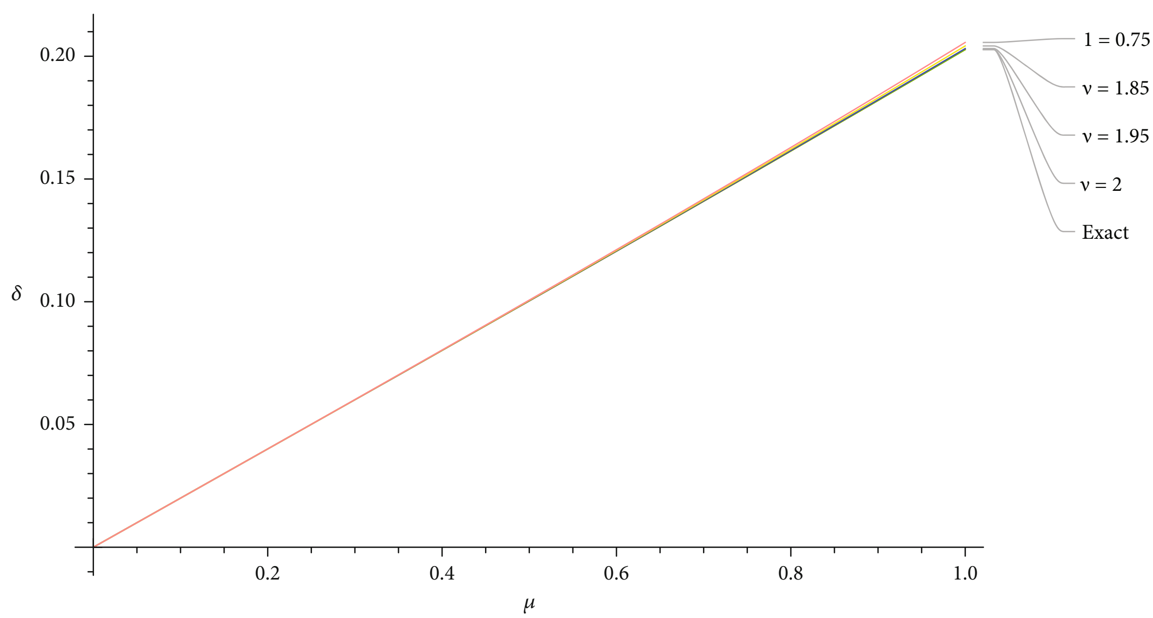
$$\begin{aligned} u(\mu,0) &= 0, \\ u_{\mu}(\mu,0) &= \mu. \end{aligned} \tag{36}$$

Taking the inverse of (ET), we get

$$u(\mu,\delta) = \mu\delta + \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[2\frac{\mu^2}{\delta}uu_{\mu}\right]\right]. \tag{37}$$



(a) 3D approximate solution Example 3



(b) 2D approximate solution Example 3

FIGURE 3: Create 2D and 3D comparison charts of precise data and predicted computational outputs for Example 3, using $\delta = 0.2$.

After matching the coefficient, we find

$$P^0 : u_0(\mu,\delta) = \mu\delta,$$

$$P^1 : u_1(\mu,\delta) = \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[2\frac{\mu^2}{\delta}H_0(u)\right]\right] = 2\mu^3\frac{\delta^{\nu+1}}{(\nu+1)!},$$

$$P^2 : u_2(\mu,\delta) = \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[2\frac{\mu^2}{\delta}H_1(u)\right]\right] = 16\mu^5\frac{\delta^{2\nu+1}}{(2\nu+1)!},$$

$$P^3 : u_3(\mu,\delta) = \mathrm{E}^{-1}\left[\beta^{\nu}\mathrm{E}\left[2\frac{\mu^2}{\delta}H_2(u)\right]\right] = \left[\frac{32\times 6}{(2\nu+1)!} + \frac{24}{(\nu+1)^2!}\right]\mu^7\frac{(2\nu+1)!\delta^{3\nu+1}}{(3\nu+1)!},$$

$$\vdots \quad \vdots \quad \vdots \tag{38}$$

The convergent solution of equation (34), is presented by

$$u(\mu, \delta) = \mu\delta + \frac{2\mu^3}{(\nu+1)!}\delta^{\nu+1} + \frac{16\mu^5}{(2\nu+1)!}\delta^{2\nu+1} + \left[\frac{32\times 6}{(2\nu+1)!} + \frac{24}{(\alpha+1)^2!}\right]\frac{(2\nu+1)!\mu^7}{(3\nu+1)!}\delta^{3\nu+1} + \cdots. \quad (39)$$

When $\nu \longrightarrow 2$, equation (39) becomes

$$u(\mu, \delta) = \mu\delta + \frac{1}{3}(\mu\delta)^3 + \frac{2}{15}(\mu\delta)^5 + \frac{17}{315}(\mu\delta)^7 + \cdots. \quad (40)$$

Therefore, we conclude that $u(\mu, \delta) = \tan(\mu\delta)$ [31]; see Table 3 and Figure 3.

In all figures, the exact and (NHPM) solutions at $nu = 1, 2$ are plotted in 2D plots of all examples, and it is observed that the exact and derived results are in good contact, confirming the proposed method's high accuracy.

4. Conclusion

The Elzaki transform method has been used with the new homotopy perturbation approach to solve nonlinear problems quickly, easily, and accurately, resulting in approximations that swiftly converge to the true answer. The strategy presented is well-suited to such problems and is quite effective. As indicated by the approximation solution's faster convergence, the relevance of ETHPM has been established.

Data Availability

On request for supporting data, the authors will provide the study's findings.

Conflicts of Interest

The authors feel that having a vie is of little use to them.

Authors' Contributions

We all looked at the document and came to an agreement on the final version.

Acknowledgments

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