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A New Type of Bivariate-Gamma Probability Function

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This paper develops a New Bivariate-Gamma Distribution (NBGD) and explores its mathematical and statistical properties such as marginal probability distributions, moments, product moment, covariance and correlation. The study further investigates the various special cases of the NBGD. The new distribution is more robust with additional parameters and flexible for modelling real datasets.

Keywords: Bivariate Γ-distribution; marginal distributions; covariance; correlation.

2010 Mathematics Subject Classification: 60E05, 62E15.

1 Introduction

The Gamma distribution is often used to describe variables bounded on one side. This density function can be adopted in analyzing the distribution of economic income, describing the sizes of grains produced in comminution, drop size distributions in sprays and so on.

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Definition 1.1. A random variable X has a generalized-gamma distribution, and is called a generalized-gamma random variable, if its probability density function is defined by

$$f(x;\alpha,\beta,\mu,c)=\begin{cases}\dfrac{1}{\beta^{\alpha\,c}\Gamma(\alpha)}c(x-\mu)^{\alpha\,c-1}e^{-\left(\frac{x-\mu}{\beta}\right)^c}; & x>0\\ 0 & \text{elsewhere}\end{cases} \tag{1.1}$$

where β (a scale parameter) and α (the shape parameter), $\mu\in\mathbb{R}$ (a location parameter) are positive real parameters, a fourth parameter c which may in principle take any real value but normally we consider the case where $c>0$ or even $c\geq 1$. Put $|c|$ in the normalization for $f(x)$ if $c<0$, and $\Gamma(.)$ is the usual Euler function defined as $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$.

Relationship of (1.1) with other distributions are:

- If $c=1$, it reduces to ordinary three-parameter gamma or shifted gamma distribution.
- If $c=1$ and $\mu=0$, a more flexible version of this distribution is obtained giving the so called two-parameters gamma distribution.
- If $c=\alpha=1$ and $\mu=0$, (1.1) reduces to exponential distribution with parameter β.
- If $\alpha=\frac{v}{2}$, $\beta=2$ and $c=1$, generalized gamma becomes chi-square distribution with parameter v. For large α it (generalized gamma distribution) is more symmetric and closer to a normal distribution.
- If $\alpha=\frac{1}{\alpha\sqrt{2}}$, $\beta=1$ and $c=2$, generalized gamma gives Rayleigh distribution with parameter α.
- If $c=1$, $\mu=0$ and $Y=\dfrac{1}{X}$ where $X\sim Gamma(\alpha,\beta)$ then $f(y)\sim IG(\alpha,\beta)$, the distribution of the reciprocal of a variable distributed according to the gamma distribution emerged which is called the inverse gamma distribution or inverted gamma distribution or the reciprocal gamma distribution.

Algebraic r^{th} moment of (1.1) is

$$E(X^r)=\int_{\mathbb{R}}x^rf(x)dx=\frac{\beta^r}{\Gamma(\alpha)}\int_0^\infty u^{\alpha+\frac{r}{c}-1}e^{-u}du=\frac{\beta^r}{\Gamma(\alpha)}\Gamma\left(\alpha+\frac{r}{c}\right) \tag{1.2}$$

For negative values of c the moments are finite for ranks v satisfying $\frac{v}{c}>-b$ (or even just avoiding the singularities $\frac{1}{\beta}+\frac{v}{c}\neq 0,-1,-2\cdots$). As expected, putting $c=1$ in (1.2) produces the r^{th} moment of shifted-gamma distribution.

The use of Generalized Gamma distribution has garnered commendable attention of many researchers in a wide spectrum of studies due to its flexibility, robustness and small magnitude of entropy. [1] (survival analysis of ovarian cancer patients), [2] (lifetimes of industrial devices, serological reversal time in children of HIV-contaminated mothers), [3] (strength of materials: breaking stress of carbon fibers, repair times for an airborne communication transceiver), [4] and [5] adopted alpha power transformed Xgamma for modelling strength of materials and environmental data, respectively, while [6] generalized conventional gamma function to study fatigue of aluminum coupons and tensile strength of carbon fibers. [7]-[8] extended generalised gamma to Xgamma and transmuted inverse Xgamma distributions with application to zoonatic disease caused by coronavirus etc.

In some experimental situations where the use of a covariate could increase precision of the experiment, the distributions of the test variate and the covariate are highly non-normal. Some of these cases can be analysed using a bivariate Γ-distribution which is discuss, modify and its mathematical characteristics explore in this work. Applications of the propose bivariate gamma probability function are to be found in wind gust, ascent flight of the space shuttle, reliability theory, signal noise, meteorology etc whch are left for future studies.

The bivariate gamma distribution is a powerful probability function that has applications in epidemiology and medical fields [9], noise theory, modeling of rainfall at two nearby rain gauges, rain-making experiments of two areas with strong correlation coefficient, the dependence between annual stream-flow of rivers and a real precipitation [10], wind gust data [11], dependence between rainfall and runoff [12], reliability theory, strength (fracture, fatigue) of different kind of materials, renewal processes and stochastic routing problems [13], and in every skewed data [14]. If $(P_i, Q_i)|i = 1, 2, \cdots, n$ is a random sample from a bivariate normal distribution with 0 means, then the bivariate random variable (X, Y), where $X = \frac{1}{n}\sum_{i=1}^{n} P_i^2$ and $Y = \frac{1}{n}\sum_{i=1}^{n} Q_i^2$, has bivariate gamma distribution. This can be advantageously proved using characteristic function, $\Phi(X, Y) = \iint e^{it(x+y)} f(x, y)dxdy$.

Definition 1.2. A continuous bivariate random variable (P, Q) is said to have the bivariate gamma distribution if its joint probability density function is of the form

$$f(p,q) = \begin{cases} \dfrac{(pq)^{\frac{1}{2}(\alpha-1)}}{(1-\theta)\Gamma(\alpha)(\theta)^{\dfrac{(\alpha-1)}{2}}} e^{-\left(\dfrac{p+q}{1-\theta}\right)} I_{\alpha-1}\left(\dfrac{2\sqrt{\theta\, pq}}{1-\theta}\right), & \text{if } 0 \leq p, q < \infty \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

where

$$I_k(t) = \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}t\right)^{k+2r}}{r!\Gamma(k+r+1)} \tag{1.4}$$

is the modified Bessel function of the first kind of order k and $\theta \in [0, 1)$ and $\alpha > 0$ are parameters. Explicitly, $f(p, q)$ is given by

$$f(p,q) = \begin{cases} \dfrac{1}{\theta^{\alpha-1}\Gamma(\alpha)} e^{-\left(\dfrac{p+q}{1-\theta}\right)} \sum_{k=0}^{\infty} \dfrac{(\theta\, pq)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}}; & \text{for } 0 \leq p, q < \infty \\ 0; & \text{otherwise.} \end{cases} \tag{1.5}$$

The main difficulty in dealing with bivariate gamma is that for many problems, no unique bivariate gamma density function exists. The definition given above is due to Kibble as cited in [15]. In the same period, Cherian developed a bivariate gamma distribution whose probability density function is given by

$$f(p,q) \begin{cases} \frac{e^{-(p+q)}}{\prod_{i=1}^{3}\Gamma(\alpha_i)} \int_0^{\min\{p,q\}} \frac{r^{\alpha_3}(p-r)^{\alpha_1}(q-r)^{\alpha_2}}{z(p-z)(q-z)} e^r dr; & \text{if } 0 < p, q < \infty \\ 0 & \text{elsewhere,} \end{cases} \tag{1.6}$$

where α_1, α_2, $\alpha_3 \in (0, \infty)$ are parameters. Mckay constructed alternative bivariate gamma distribution whose probability density function is of the form

$$f(p,q) \begin{cases} \frac{\theta^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(q-p)^{\beta-1} e^{-\theta\, q}; & \text{if } 0 < p < q < 0 \\ 0; & \text{elsewhere} \end{cases} \tag{1.7}$$

where $\theta, \alpha, \beta \in (0, \infty)$ are constants. Equations (1.6) and (1.7) are all cited in [15]. [16] have studied a density function of the form,

$$f(x; \alpha, \beta, \delta, \alpha, \delta) = \frac{\beta\alpha^{\frac{m}{\beta}+1}}{\nu_X(0)} x^{\beta+m-1} e^{-\delta\, x^{\beta}} 2F_1\left(\lambda, b; c - \frac{\alpha\, x^{\beta}}{n}\right) I_{\mathbb{R}} + (x), \tag{1.8}$$

where $\beta \geq 0$, $m+\beta > 0$, δ and α are positive numbers, and $\nu_X(0)$ is generally defined as

$$\nu_X(h) = \alpha^{\frac{m}{\beta}+1}\Gamma(c) \left[\begin{array}{l} \frac{\delta^{-\frac{m+h-b\beta+\beta}{\beta}}\Gamma\left(\frac{m+h-b\beta+\beta}{\beta}\right)\Gamma(\lambda-b)\alpha^{-b}}{\Gamma(c-b)\Gamma(\lambda)} \times {}_2F_2\left(b, b-c+1; b-\frac{m+h}{\beta}, b-\lambda+1; \frac{\delta}{\alpha}\right) + \\ \left(\frac{\Gamma\left(\frac{m+h+\beta}{\beta}\right)\Gamma\left(-\frac{m+h-b\beta+\beta}{\beta}\right)\Gamma\left(-\frac{m+h+\beta-\beta\lambda}{\beta}\right)\alpha^{-\frac{m+h+\beta}{\beta}}}{\Gamma\left(-\frac{m+h-c\beta+\beta}{\beta}\right)\Gamma(\lambda)\Gamma(b)} \times {}_2F_2(K)\right) + \\ \frac{\delta^{-\frac{m+h+\beta-\beta\lambda}{\beta}}\Gamma(b-\lambda)\Gamma\left(\frac{m+h+\beta-\beta\lambda}{\beta}\right)\alpha^{-\lambda}}{\Gamma(b)\Gamma(c-\lambda)} \times {}_2F_2 \\ \left(\lambda, -c+\lambda+1; -b+\lambda+1, \lambda-\frac{m+h}{\beta}; \frac{\delta}{\alpha}\right) \end{array} \right] \tag{1.9}$$

such that $K = \frac{m+h}{\beta}+1, -c+\frac{m+h}{\beta}+2; -b+\frac{m+h}{\beta}+2, -\lambda+\frac{m+h}{\beta}+2; \frac{\delta}{\alpha}$ for $h = 0, 1, \cdots$, whenever $h+m+\beta > 0$. The h^{th} moment of this distribution is

$$\mu'_X(h) = \frac{\nu_X(h)}{\nu_X(0)}, \tag{1.10}$$

A multivariate case of bivariate gamma distribution was developed by [13], and [10].

From (1.5), results show that the marginal distributions of P and Q, $g(p) = \frac{1}{\Gamma(\alpha)}p^{\alpha-1}; p > 0$ and $g(q) = \frac{1}{\Gamma(\alpha)}q^{\alpha-1}; q > 0$ are univariate gamma with parameter α (and $\theta = 1$). This study modifies and reparameterizes Kibble bivariate gamma density function with the aim of making $\theta \in \mathbb{R}$ (other than fixed value $\theta = 1$) in the marginal pdf's and other accompanies probability properties.

2 Preliminaries

The study states the following lemmas without proof since they have been proved in many elementary Calculus textbooks.

Lemma 2.1. *From Calculus,*

$$e^t = \sum_{i=0}^{\infty} \frac{t^i}{i!}. \tag{2.1}$$

The infinite series on RHS of (2.1) converges $\forall t \in \mathbb{R}$. *Differentiating both sides of (2.1) wrt* t *and then multiplying the resulting expression by* t *gives*

$$te^t = \sum_{i=0}^{\infty} i\frac{t^i}{i!} \tag{2.2}$$

Differentiating (2.2) again and multiply the resulting expression by t *to get*

$$te^t + t^2e^t = \sum_{i=0}^{\infty} i^2\frac{t^i}{i!} \tag{2.3}$$

3 Results

The study begins by letting $p = \theta x$ and $q = \theta y$ with the Jacobian coefficient of the transformation as

$$J = \begin{vmatrix} \frac{dp}{dx} & \frac{dp}{dy} \\ \frac{dq}{dx} & \frac{dq}{dy} \end{vmatrix} = \begin{vmatrix} \theta & 0 \\ 0 & \theta \end{vmatrix} = \theta^2. \tag{3.1}$$

Suppose there exists, a one to one transformation that maps set $A = \{(p, q) : 0 \leq p \leq \infty, \quad 0 \leq q \leq \infty\}$ onto $B = \{(x, y) : 0 \leq x \leq \infty, \quad 0 \leq y \leq \infty\}$. Hence, from (1.5), the joint pdf of X and Y, say $f(x, y) = f(p = \theta x, q = \theta y) \, |J|$, is

$$f(x,y) = \begin{cases} \dfrac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\dfrac{\theta}{1-\theta}(x+y)} \sum\limits_{k=0}^{\infty} \dfrac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}}; & \text{for } 0 \leq x, y < \infty \\ 0; & \text{otherwise.} \end{cases} \tag{3.2}$$

where $\theta \in (0, \infty)$ and $\alpha > 0$ are real constants (parameters).

Theorem 3.1. *The developed BGD in (3.2) is a proper pdf.*

Proof. This is to show that $\iint f(x, y)dxdy = 1$. Following from (3.2),

$$\begin{aligned}
\int_0^{\infty}\int_0^{\infty} f(x,y)dxdy &= \int_0^{\infty}\int_0^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{\frac{-\theta}{1-\theta}(x+y)} \sum_{k=0}^{\infty} \frac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} dydx \\
&= \int_0^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{(\theta^3 x)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^{\infty} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} dydx \\
&= \int_0^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{(\theta^3 x)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \Gamma(\alpha+k)\left(\frac{1-\theta}{\theta}\right)^{\alpha+k} dx \\
&= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+2k}}{k!(1-\theta)^k} x^{\alpha+k-1} dx \\
&= \int_0^{\infty} \frac{\theta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k dx
\end{aligned}$$

Using (2.1), $\sum\limits_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k = e^{\dfrac{\theta^2 x}{1-\theta}}$,

$$\begin{aligned}
\int_0^{\infty}\int_0^{\infty} f(x,y)dxdy &= \int_0^{\infty} \frac{\theta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\theta}{1-\theta}x} e^{\frac{\theta^2}{1-\theta}x} dx \\
&= \frac{\theta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-\theta x} dx \\
&= \frac{\theta^{\alpha}}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha)}{\theta^{\alpha}}\right) \\
&= 1.
\end{aligned}$$

□

3.1 The Marginal Distributions and Joint Expectation

Theorem 3.2. *The marginal distributions of X and Y are each univariate gamma with parameter α and θ. That is,*

$$\begin{aligned}
f_1(x) &= \int f(x,y)dy = \frac{\theta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x} \sim GAM(x;\alpha,\theta), \quad \text{and} \\
f_2(y) &= \int f(x,y)dx = \frac{\theta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta y} \sim GAM(y;\alpha,\theta).
\end{aligned} \tag{3.3}$$

Proof. This requires integrating (3.2) wrt y. That is,

$$
\begin{aligned}
f_1(x) = \int f(x,y)dy &= \int_0^\infty \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}(x+y)} \sum_{k=0}^{\infty} \frac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} dy \\
&= \sum_{k=0}^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \frac{(\theta^3 x)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^\infty y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} dy \\
&= \sum_{k=0}^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \frac{(\theta^3 x)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \left(\frac{1-\theta}{\theta}\right)^{\alpha+k} \Gamma(\alpha+k) \\
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \frac{\theta^{\alpha+2k}}{k!(1-\theta)^k} x^{\alpha+k-1} \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\theta}{1-\theta}x} e^{\frac{\theta^2}{1-\theta}x} \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x} \sim GAM(x;\alpha,\theta)
\end{aligned}
$$

Similarly, as shown below, the marginal distribution of Y is also univariate gamma.

$$
\begin{aligned}
f_2(y) = \int f(x,y)dx &= \sum_{k=0}^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \frac{(\theta^3 y)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^\infty x^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}x} dx \\
&= \sum_{k=0}^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \frac{(\theta^3 y)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \left(\frac{1-\theta}{\theta}\right)^{\alpha+k} \Gamma(\alpha+k) \\
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \frac{\theta^{\alpha+2k}}{k!(1-\theta)^k} y^{\alpha+k-1} \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\theta}{1-\theta}y} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 y}{1-\theta}\right)^k \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\theta}{1-\theta}y} e^{\frac{\theta^2 y}{1-\theta}} \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta y} \sim GAM(y;\alpha,\theta)
\end{aligned}
$$

□

From the two marginals,

$$
\begin{aligned}
E(X) &= \int_0^\infty x f_1(x)dx = \int_0^\infty x \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x} dx = \frac{\theta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} = \frac{\alpha}{\theta} \\
E(Y) &= \int_0^\infty y f_2(y)dy = \int_0^\infty y \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta y} dy = \frac{\theta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} = \frac{\alpha}{\theta}
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
Var(X) &= \int_0^\infty x^2 \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x} dx - \left[\int_0^\infty x \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x} dx\right]^2 = \frac{\theta^\alpha \alpha(\alpha+1)}{\theta^{\alpha+2}} - \left[\frac{\alpha}{\theta}\right]^2 = \frac{\alpha}{\theta^2} \\
Var(Y) &= \int_0^\infty y^2 \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta y} dy - \left[\int_0^\infty y \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta y} dy\right]^2 = \frac{\theta^\alpha \alpha(\alpha+1)}{\theta^{\alpha+2}} - \left[\frac{\alpha}{\theta}\right]^2 = \frac{\alpha}{\theta^2}
\end{aligned} \tag{3.5}
$$

The joint expectation is

$$\begin{aligned}
E(XY) &= \iint xyf(x,y)dxdy = \int_0^\infty \int_0^\infty xy \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}(x+y)} \sum_{k=0}^\infty \frac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} dxdy \\
&= \int_0^\infty y \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \sum_{k=0}^\infty \frac{(\theta^3 y)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^\infty x^{\alpha+k} e^{-\frac{\theta}{1-\theta}x} dxdy \\
&= \int_0^\infty y \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \sum_{k=0}^\infty \frac{(\theta^3 y)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \Gamma(\alpha+k+1) \left(\frac{1-\theta}{\theta}\right)^{\alpha+k+1} dy \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{\theta^{\alpha+2k-1}}{k!(1-\theta)^{k-1}} e^{-\frac{\theta}{1-\theta}y} y^{\alpha+k}(\alpha+k)dy \\
&= \frac{\alpha}{\Gamma(\alpha)} \int_0^\infty \sum_{k=0}^\infty \frac{\theta^{\alpha+2k-1}}{k!(1-\theta)^{k-1}} e^{-\frac{\theta}{1-\theta}y} y^{\alpha+k} dy + \frac{1}{\Gamma(\alpha)} \int_0^\infty \sum_{k=0}^\infty \frac{k\theta^{\alpha+2k-1}}{k!(1-\theta)^{k-1}} e^{-\frac{\theta}{1-\theta}y} y^{\alpha+k} dy \\
&= \frac{\alpha(1-\theta)\theta^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y^\alpha e^{-\frac{\theta}{1-\theta}y} \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{\theta^2 y}{1-\theta}\right)^k dy + \frac{(1-\theta)\theta^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y^\alpha e^{\frac{-\theta}{1-\theta}y} \sum_{k=1}^\infty \frac{k}{k!} \left(\frac{\theta^2 y}{1-\theta}\right)^k dy
\end{aligned}$$

Using (2.1) and (2.2), $\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{\theta^2 y}{1-\theta}\right)^k = e^{\frac{\theta^2}{1-\theta}y}$ and $\sum_{k=0}^\infty \frac{k}{k!}\left(\frac{\theta^2 y}{1-\theta}\right)^k = \frac{\theta^2 y}{1-\theta} e^{\frac{\theta^2}{1-\theta}y}$. So,

$$\begin{aligned}
E(XY) &= \frac{\alpha(1-\theta)\theta^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y^\alpha e^{-\theta y} dy + \frac{\theta^{\alpha+1}}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-\theta y} dy \\
&= \frac{\alpha(1-\theta)\theta^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} + \frac{\theta^{\alpha+1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}} \qquad (3.6) \\
&= \frac{\alpha^2(1-\theta)}{\theta^2} + \frac{\alpha(\alpha+1)}{\theta}
\end{aligned}$$

By definition, $Cov(X,Y) = E(XY) - E(X)E(Y)$. Hence,

$$Cov(X,Y) = \frac{\alpha^2(1-\theta)}{\theta^2} + \frac{\alpha(\alpha+1)}{\theta} - \left(\frac{\alpha}{\theta}\right)\left(\frac{\alpha}{\theta}\right) = \frac{\alpha}{\theta} \qquad (3.7)$$

The correlation coefficient of X and Y can be easily computed using (3.5) and (3.7) as

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\alpha}{\theta} \div \left(\frac{\alpha}{\theta^2}\right) = \theta. \qquad (3.8)$$

3.2 The Conditional Mean and Vaariance

Theorem 3.3. *Let the random variable* $(X,Y) \sim K(\alpha,\theta)$, *where* $0 < \alpha < \infty$ *and* $0 \le \theta < 1$. *Then*

$$E(Y/x) = \frac{\alpha(1-\theta)}{\theta} + \theta x \qquad (3.9)$$

$$E(X/y) = \frac{\alpha(1-\theta)}{\theta} + \theta y \qquad (3.10)$$

$$Var(Y/x) = \frac{(1-\theta)}{\theta^2}\left[2\theta^2 x + \alpha(1-\theta)\right] \qquad (3.11)$$

$$Var(X/y) = \frac{(1-\theta)}{\theta^2}\left[2\theta^2 y + \alpha(1-\theta)\right] \qquad (3.12)$$

Proof. First, the conditional probability density function Y given $X = x$, is given by

$$f(y/x) = \frac{f(x,y)}{f_1(x)} = \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}(x+y)} \sum_{k=0}^{\infty} \frac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \times \frac{\Gamma(\alpha)}{\theta^{\alpha} x^{\alpha-1} e^{-\theta x}}$$

$$= e^{-\frac{\theta}{1-\theta}x+\theta x} \sum_{k=0}^{\infty} \frac{\theta^{3\alpha+3k-3-\alpha+3-\alpha} x^{\alpha+k-1-\alpha+1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y}$$

$$= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y}$$

Then, the conditional mean is

$$E(Y|\,X=x) = \int_0^{\infty} y f(y/x) dy = \int_0^{\infty} y e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} dy$$

$$= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^{\infty} y^{\alpha+k} e^{-\frac{\theta}{1-\theta}y} dy$$

$$= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \frac{\Gamma(\alpha+k+1)}{\left(\frac{\theta}{1-\theta}\right)^{\alpha+k+1}}$$

$$= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{2k-1} x^k}{k!(1-\theta)^{k-1}} (\alpha+k)$$

$$= \frac{1-\theta}{\theta} e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k (\alpha+k)$$

$$= \frac{1-\theta}{\theta} e^{-\frac{\theta^2}{1-\theta}x} \left[\alpha \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k + \frac{\theta^2 x}{1-\theta} \sum_{k-1=0}^{\infty} \frac{1}{(k-1)!} \left(\frac{\theta^2 x}{1-\theta}\right)^{k-1}\right]$$

From (2.1), $\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k = \sum_{k-1=0}^{\infty} \frac{1}{(k-1)!} \left(\frac{\theta^2 x}{1-\theta}\right)^{k-1} = e^{-\frac{\theta^2}{1-\theta}x}$. Therefore, the conditional mean, $E(Y|\,X=x)$, is finally presented as

$$E(Y|\,X=x) = \frac{1-\theta}{\theta} e^{-\frac{\theta^2}{1-\theta}x} \left[\alpha\, e^{\frac{\theta^2}{1-\theta}x} + \frac{\theta^2 x}{1-\theta} e^{\frac{\theta^2}{1-\theta}x}\right] = \frac{\alpha(1-\theta)}{\theta} + \theta\, x. \tag{3.13}$$

In the same manner,

$$E(Y^2|\,X=x) = \int_0^{\infty} y^2 f(y/x) dy = \int_0^{\infty} y^2 e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} dy$$

$$= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^{\infty} y^{\alpha+k+1} e^{-\frac{\theta}{1-\theta}y} dy$$

$$= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \frac{\Gamma(\alpha+k+2)}{\left(\frac{\theta}{1-\theta}\right)^{\alpha+k+2}}$$

$$= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \frac{(\alpha+k)(\alpha+k+1)\Gamma(\alpha+k)(1-\theta)^{\alpha+k+1}}{\theta^{\alpha+k+2}}$$

$$= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{2k-2} x^k}{k!(1-\theta)^{k-2}} (\alpha+k)(\alpha+k+1)$$

which gives

$$\begin{aligned} E(Y^2|\, X=x) &= \frac{(1-\theta)^2}{\theta^2} e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k (\alpha+k)[(\alpha+1)+k] \\ &= \frac{(1-\theta)^2}{\theta^2} e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k [\alpha(\alpha+1)+(2\alpha+1)k+k^2] \end{aligned} \tag{3.14}$$

From (2.1), (2.2) and (2.3), $\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k$, $\sum_{k=0}^{\infty}\frac{k}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k$ and $\sum_{k=0}^{\infty}\frac{k^2}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k$ are $e^{\frac{\theta^2 x}{1-\theta}}$, $\frac{\theta^2 x}{1-\theta}e^{\frac{\theta^2 x}{1-\theta}}$ and $\frac{\theta^2 x}{1-\theta}e^{\frac{\theta^2}{1-\theta}x}+\left(\frac{\theta^2 x}{1-\theta}\right)^2 e^{\frac{\theta^2 x}{1-\theta}}$, respectively. Hence, (3.14) gives

$$\begin{aligned} E(Y^2|\, X=x) &= \frac{(1-\theta)^2}{\theta^2} e^{-\frac{\theta^2}{1-\theta}x} \left[\alpha(\alpha+1)e^{\frac{\theta^2}{1-\theta}x} + (2\alpha+1)\frac{\theta^2 x}{1-\theta}e^{\frac{\theta^2}{1-\theta}x} + \left(\frac{\theta^2 x}{1-\theta}\right)e^{\frac{\theta^2}{1-\theta}x} + \left(\frac{\theta^2 x}{1-\theta}\right)^2 e^{\frac{\theta^2}{1-\theta}x}\right] \\ &= \frac{(1-\theta)^2}{\theta^2}\left[\alpha(\alpha+1) + (2\alpha+1)\frac{\theta^2 x}{1-\theta} + \frac{\theta^2 x}{1-\theta} + \left(\frac{\theta^2 x}{1-\theta}\right)^2\right] \\ &= \frac{\alpha(\alpha+1)(1-\theta)^2}{\theta^2} + 2(\alpha+1)(1-\theta)x + \theta^2 x^2 \end{aligned} \tag{3.15}$$

The conditional variance of Y given $X = x$ is

$$\begin{aligned} Var(Y|\, x) &= E(Y^2|\, X=x) - [E(Y|\, X=x)]^2 \\ &= \frac{\alpha(\alpha+1)(1-\theta)^2}{\theta^2} + 2(\alpha+1)(1-\theta)x + \theta^2 x^2 - \left[\frac{\alpha(1-\theta)}{\theta} + \theta\, x\right]^2 \\ &= \frac{\alpha(1-\theta)^2}{\theta^2} + 2(1-\theta)x \\ &= \frac{(1-\theta)}{\theta^2}\left[2\theta^2 x + \alpha(1-\theta)\right]. \end{aligned} \tag{3.16}$$

The developed density function (3.2) is symmetric, that is, $f(x,y) = f(y,x)$. As a result, $E(X|\, y)$ and $Var(X|\, y)$ can be obtained by interchanging x with y in (3.13) and (3.16), respectively. So,

$$E(X/y) \quad = \quad \frac{\alpha(1-\theta)}{\theta} + \theta\, y \tag{3.17}$$

$$Var(X/y) \quad = \quad \frac{(1-\theta)}{\theta^2}\left[2\theta^2 y + (1-\theta)\right] \tag{3.18}$$

This completes the proof. □

Theory establishes that the univariate exponential distribution is a special case of the univariate gamma distribution. Similarly, the bivariate exponential distribution is a special case of bivariate gamma distribution [17]. Taking the index parameter to be unity in the developed bivariate gamma density function presented in (3.2), we obtain its corresponding bivariate exponential distribution as

$$f(x,y) = \begin{cases} \theta^2 e^{-\frac{\theta}{1-\theta}(x+y)} \sum_{k=0}^{\infty} \frac{(\theta^3 xy)^k}{k!\Gamma(k+1)(1-\theta)^{2k+1}}; & \text{for } 0 \le x, y < \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

where $\theta \in (0, \infty)$ is a parameter. In addition, each of the properties of bivariate gamma distribution is same with bivariate exponential distribution with $\alpha = 1$. The equivalent bivariate exponential probability density function corresponding to bivariate gamma distribution of Kibble is

$$f(p,q) = \begin{cases} e^{-\left(\frac{p+q}{1-\theta}\right)} \sum\limits_{k=0}^{\infty} \frac{(\theta\, pq)^k}{k!\Gamma(k+1)(1-\theta)^{2k+1}}; & \text{if } 0 < p, q < \infty \\ 0; & \text{otherwise} \end{cases} \tag{3.20}$$

where $\theta \in (0, 1)$ is a parameter. The bivariate exponential distribution corresponding to the Cherian bivariate distribution is

$$f(p,q) = \begin{cases} \left[e^{\min\{p,q\}} - 1\right] e^{-(p+q)}; & \text{if } 0 < p, q < \infty \\ 0; & \text{elsewhere} \end{cases} \tag{3.21}$$

Gumble, cited in [15], constructed a bivariate exponential density function of the form

$$f(p,q) = \begin{cases} [(1+\theta\, p)(1+\theta\, q) - \theta]\, e^{-(p+q+\theta\, pq)}; & \text{if } 0 < p, q < \infty \\ 0; & \text{otherwise} \end{cases} \tag{3.22}$$

and $\theta > 0$ is a parameter. Marshall and Olkin, as cited in [[15]], introduced alternative bivariate exponential distribution as

$$f(p,q) = \begin{cases} 1 - e^{(\alpha+\gamma)p} - e^{-(\beta+\gamma)q} + e^{-(\alpha\, p+\beta\, q+\gamma\max\{p,q\})}; & \text{if } p, q > 0 \\ 0; & \text{otherwise} \end{cases} \tag{3.23}$$

where $\alpha, \beta, \gamma > 0$ are unknown but fixed parameters. (3.23) satisfies the memoryless property.

4 Conclusion

The study developed a new probability function, namely, modified bivariate gamma probability distribution function. The main properties (the marginal densities, moments, product moment, covariance and correlation between the two variables X and Y) of the developed probability function were derived. The re-parameterized distribution has the advantage of robustness, simple implementation and opportunity of varying values of θ for mixed outcome data.

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Conflict of Interest

Authors declare that there is no conflict of interest.

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