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Resolution of the Standard Telegraph Equation by the Laplace-Adomian Method

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we resarch the solution of the standard telegraph equation by the Laplace-Adomian method. The Laplace-Adomian method is based on the combination of Laplace transform and the Adomian decompositionnal method.

Keywords: Telegraph equation; Laplace transform; ADM method.

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1 Introduction

In this article, we study the general solution of the standard telegraph equations by the method of Laplace-Adomian. The standard telegraph equation is an important equation arises in the propagation of electrical signals along a telegraph line, taking into consideration the inductance, capacitance and conductance of the cable [1, 2, 3, 4]. However the method of Laplace-Adomian is a numerical method based on the combination of the Laplace Transform and Adomian decompositionnal method [5, 6, 2].

2 The numerical Laplace-Adomian method

The standard telegraph equation is a partial differential equation given by [2] :

$$\frac{\partial^2 u}{\partial x^2} = \alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u$$

where $u = u(t, x)$ is the resistance, and α,β and γ are constants related to the inductance, capacitance and conductance of the cable respectively.

Let us consider the following functional equation:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} &= \alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u \\ u(t,0) &= f(t) \\ \frac{\partial u}{\partial x}(t,0) &= g(t) \\ u(0,x) &= h(x) \\ \frac{\partial u}{\partial t}(0,x) &= v(x) \end{cases} \tag{1}$$

Taking $Lu = \frac{\partial^2 u}{\partial x^2}$ and $Ru = \alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u$

We have :

$$Lu = Ru \tag{2}$$

Where L is an invertible operator in the Adomian sense and R the linear remainder.

Applying the laplace transform to the equation (1), we obtain :

$$\mathcal{L}_x(Lu) = \mathcal{L}_x(Ru) \Leftrightarrow p^2\mathcal{L}_x(u) - pu(t,0) - \frac{\partial u}{\partial x}(t,0) = \mathcal{L}_x(Ru) \tag{3}$$

$$p^2\mathcal{L}_x(u) = pf(t) + g(t) + \mathcal{L}_x(Ru) \tag{4}$$

Using the decomposition series for the linear term $u(t, x)$ gives

$$p^2\sum_{n\geq 0}\mathcal{L}_x(u_n) = pf(t) + g(t) + \sum_{n\geq 0}\mathcal{L}_x(Ru_n) \tag{5}$$

This yields the following Adomian algorithm:

$$\begin{cases} p^2\mathcal{L}_x(u_0) = pf(t) + g(t) \\ p^2\mathcal{L}_x(u_{n+1}) = \mathcal{L}_x(Ru_n); n \geq 0 \end{cases} \tag{6}$$

Applying the laplace transform to the equation (2), we obtain :

$$\begin{cases} u_0(t,x) = \mathcal{L}_x^{-1}\left[\frac{1}{p^2}(pf(t)+g(t))\right] \\ \quad\quad\quad \cdots \\ u_{n+1}(t,x) = \mathcal{L}_x^{-1}\left[\frac{1}{p^2}\mathcal{L}_x(Ru_n)\right]; n \geq 0 \end{cases} \tag{7}$$

3 Algorithm of Laplace - ADM Convergence's

Considering the equation (1)

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} & = & \alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u \\ u(t,0) & = & f(t) \\ \frac{\partial u}{\partial x}(t,0) & = & g(t) \\ u(0,x) & = & h(x) \\ \frac{\partial u}{\partial t}(0,x) & = & v(x) \end{cases}$$

With $(t,x) \in \Omega$ where $\Omega = [0;+\infty[\times [a,b]$

The application of the Laplace-ADM method gives

$$\begin{cases} u_0(t,x) = \mathcal{L}_x^{-1}\left[\frac{1}{p^2}(pf(t)+g(t))\right] \\ u_{n+1}(t,x) = \mathcal{L}_x^{-1}\left[\frac{1}{p^2}\mathcal{L}_x(Ru_n)\right]; n \geq 0 \end{cases}$$

Let us suppose :

· (H_1)

f is continuous then there is a real M so that

$$|f(t)| \leq M \text{ for all } t \in [0,T]$$

· (H_2)

g is continuous then there is a real M so that

$$|g(t)| \leq M' \text{ for all } t \in [0,T]$$

However

R the linear remainder is continuous then there is a real $\lambda > 0$ so that

$$||Ru|| \leq \lambda\,||u||$$

Indeed, we have :

$$\begin{cases} |u_0| & = & \left|\mathcal{L}_x^{-1}\left[\dfrac{f(t)}{p}\right] + \mathcal{L}_x^{-1}\left[\dfrac{g(t)}{p^2}\right]\right| \\ ... & ... & ... \\ |u_n| & = & \left|\mathcal{L}_x^{-1}\left[\dfrac{\mathcal{L}_t(Ru_{n-1})}{p}\right]\right| \ ; n \geq 1 \end{cases}$$

There is a real $x_0 \in \mathbb{R}_*^+$ so that $\Re e(p) > x_0$, we deduce the following system :

$$\begin{cases} |u_0| & \leq & \left|\mathcal{L}_x^{-1}\left[\dfrac{f(t)}{p}\right]\right| + \left|\mathcal{L}_x^{-1}\left[\dfrac{g(t)}{p^2}\right]\right| \\ ... & ... & ... \\ |u_n| & \leq & \mathcal{L}_x^{-1}\left[\dfrac{|Ru_{n-1}|}{|p^2|}\right] \ ; n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ |u_n| & \leq & \mathcal{L}_x^{-1}\left[\dfrac{\mathcal{L}_x(|Ru_{n-1}|)}{x_0^2}\right] \ ; n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ |u_n| & \leq & \dfrac{1}{x_0^2}\mathcal{L}_x^{-1}\left[\mathcal{L}_x\left(\left|Ru_{n-1}^1\right|\right)\right] \ ; n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ |u_n| & \leq & \dfrac{\lambda}{x_0^2}\left|\left|u_{n-1}^1\right|\right| \ ; n \geq 1 \end{cases}$$

Step by step, we deduce :

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ |u_n| & \leq & \left(\dfrac{\lambda}{x_0^2}\right)^n (M + M'b) \ ; n \geq 1 \end{cases}$$

With $\dfrac{\lambda}{x_0^2} < 1$ and $x_0 \neq \sqrt{\lambda}$, we obtain

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ \sum_{n\geq 0} |u_n| & \leq & \dfrac{(M + M'b)\, x_0^2}{x_0^2 - \lambda} \end{cases}$$

Then the series $\sum_{n\geq 0} u_n$ is convergent, therefore this algorithm is convergent.

4 Applications

4.1 Example : wave equation of microstrip antenna

In this example, we study the standard linear telegraph equation presented in wave equation of microstrip antenna equation is given as [7]

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial t^2} + 2\dfrac{\partial u}{\partial t} - u & = & 0 \\ \dfrac{\partial u}{\partial x}(t,0) & = & e^{-2t} \\ u(t,0) & = & e^{-2t} \end{cases}$$

Taking $Lu = \dfrac{\partial^2 u}{\partial x^2}$, $Ru = -\dfrac{\partial^2 u}{\partial t^2} - 2\dfrac{\partial u}{\partial t} + u$.

Where L is an invertible operator in the Adomian sense and R the linear remainder.

Applying the laplace transform to the equation (2), we obtain :

$$\mathcal{L}_x(Lu) = \mathcal{L}_x(Ru) \tag{8}$$

$\Longleftrightarrow$

$$p^2\mathcal{L}_x(u) - pu(t,0) - \frac{\partial u}{\partial x}(t,0) = \mathcal{L}_x\left(-\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} + u\right) \tag{9}$$

$$\left(p^2 - 1\right)\mathcal{L}_x(u) = pe^{-2t} + e^{-2t} + \mathcal{L}_x\left(-\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}\right) \tag{10}$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$\left(p^2 - 1\right)\sum_{n\geq 0}\mathcal{L}_x(u_n) = pe^{-2t} + e^{-2t} + \sum_{n\geq 0}\mathcal{L}_x\left(-\frac{\partial^2 u_n}{\partial t^2} - 2\frac{\partial u_n}{\partial t}\right) \tag{11}$$

We deduce the following Laplace-Adomian algorithm

$$\begin{cases} \left(p^2 - 1\right)\mathcal{L}_x(u_0) = pe^{-2t} + e^{-2t} \\ \left(p^2 - 1\right)\mathcal{L}_x(u_{n+1}) = \mathcal{L}_x(Ru_n)\,; n \geq 0 \end{cases} \tag{12}$$

We obtain

$$\begin{cases} u_0(t,x) = \mathcal{L}_x^{-1}\left[\dfrac{1}{(p^2-1)}\left(pe^{-2t} + e^{-2t}\right)\right] \\ u_{n+1}(t,x) = \mathcal{L}_x^{-1}\left[\dfrac{1}{(p^2-1)}\mathcal{L}_x(Ru_n)\right]; n \geq 0 \end{cases}$$

Determinate $u_n(t,x)$, **for** $n \geq 0$

$u_0(t,x) = \mathcal{L}_x^{-1}\left[\dfrac{1}{(p^2-1)}(p+1)\,e^{-2t}\right]$

$\Rightarrow u_0(t,x) = \mathcal{L}_x^{-1}\left[\frac{1}{(p-1)}e^{-2t}\right]$

$\Rightarrow u_0(t,x) = e^{x-2t}$

$u_1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{(p^2-1)}\mathcal{L}_t\left(R(u_0)\right)\right]$

$\Rightarrow u_1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{(p^2-1)}\left(-4e^{x-2t}+4e^{x-2t}\right)\right]$

$\Rightarrow u_1(t,x) = 0$

$u_2(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{(p^2-1)}\mathcal{L}_t\left(R(u_1)\right)\right]$

$\Rightarrow u_2(t,x) = 0$

In recursive way, we deduce

$$u_n\left(t,x\right) = 0 \text{ for all } n \geq 1$$

Then

$$u(t,x) = \sum_{n\geq 0} u_n(t,x) = e^{x-2t}$$

The exact solution of model is

$$u(t,x) = e^{x-2t}$$

4.2 Example : wave equation of microstrip antenna

In this example, we study the standard linear telegraph equation is given as [7, 8, 9, 10, 4]

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} & = & \frac{\partial^2 u}{\partial t^2} + 4\frac{\partial u}{\partial t} + 4u \\ \frac{\partial u}{\partial t}\left(0,x\right) & = & -2 \\ u\left(0,x\right) & = & 1+e^{2x} \end{cases}$$

Taking $Lu = \frac{\partial^2 u}{\partial t^2}$, $Ru = \frac{\partial^2 u}{\partial x^2} - 4\frac{\partial u}{\partial t} - 4u$.

Where L is an invertible operator in the Adomian sense and R the linear remainder.

Applying the laplace transform to the equation (2), we obtain :

$$\mathcal{L}_t\left(Lu\right) = \mathcal{L}_t\left(Ru\right) \Leftrightarrow p^2\mathcal{L}_t\left(u\right) - pu\left(0,x\right) - \frac{\partial u}{\partial x}\left(0,x\right) = \mathcal{L}_t\left(\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial u}{\partial t} - 4u\right) \quad (13)$$

$$p^2\mathcal{L}_x\left(u\right) = p\left(1+e^{2x}\right) - 2 + \mathcal{L}_x\left(\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial u}{\partial t} - 4u\right) \tag{14}$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$p^2\sum_{n\geq 0}\mathcal{L}_x\left(u_n\right) = p - 2 + pe^{2x} + \sum_{n\geq 0}\mathcal{L}_x\left(\frac{\partial^2 u_n}{\partial x^2} - 4\frac{\partial u_n}{\partial t} - 4u\right) \tag{15}$$

$$p^2\sum_{n\geq 0}\mathcal{L}_x\left(u_n\right) = p - 2 + pe^{2x} + \sum_{n\geq 0}\mathcal{L}_x\left(\frac{\partial^2 u_n}{\partial x^2} - 4\frac{\partial u_n}{\partial t} - 4u\right) \tag{16}$$

We deduce the following Laplace-Adomian algorithm

$$\begin{cases} p^2\mathcal{L}_x\left(u_0\right) = p - 2 + pe^{2x} \\ p^2\mathcal{L}_x\left(u_{n+1}\right) = \mathcal{L}_x\left(Ru_n\right); n\geq 0 \end{cases} \tag{17}$$

We obtain

$$\begin{cases} u_0\left(t,x\right) = \mathcal{L}_x^{-1}\left[\frac{1}{p^2}\left(p-2+pe^{2x}\right)\right] \\ u_{n+1}\left(t,x\right) = \mathcal{L}_x^{-1}\left[\frac{1}{p^2}\mathcal{L}_x\left(Ru_n\right)\right]; n\geq 0 \end{cases}$$

Determinate $u_n(t,x)$, **for** $n\geq 0$

$u_0(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\left(p-2+pe^{2x}\right)\right]$

$\Rightarrow u_0(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p}\left(1+e^{2x}\right) - \frac{2}{p^2}\right]$

$\Rightarrow u_0(t,x) = e^{2x} - 2t + 1$

$u_1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\mathcal{L}_t\left(R(u_0)\right)\right]$

$\Rightarrow u_1(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\left[\mathcal{L}_t\left(-8t+4\right)\right]\right]$

$\Rightarrow u_1(t,x) = \mathcal{L}_t^{-1}\left(-\frac{8}{p^4} + \frac{4}{p^3}\right) = -\frac{8t^3}{3!} + \frac{4t}{2!}$

$\Rightarrow u_1(t,x) = -\frac{(2t)^3}{3!} + \frac{(2t)^2}{2!}$

$u_2(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\mathcal{L}_t\left(R(u_1)\right)\right]$

$\Rightarrow u_2(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\left[\mathcal{L}_t\left(-16t - 24t^2 - 32\frac{t^3}{3!}\right)\right]\right]$

$\Rightarrow u_2(t,x) = \mathcal{L}_t^{-1}\left(-16\frac{1}{p^4} - 48\frac{1}{p^5} - 32\frac{1}{p^6}\right)$

$\Rightarrow u_2(t,x) = -2\frac{(2t)^3}{3!} - 3\frac{(2t)^4}{4!} - \frac{(2t)^5}{5!}$

$u_3(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\mathcal{L}_t\left(R(u_2)\right)\right]$

$\Rightarrow u_3(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\left[\mathcal{L}_t\left(2^6\frac{t^2}{2!} + 2^8\frac{t^3}{3!} + 10\times 2^5\frac{t^4}{4!} + 2^7\frac{t^5}{5!}\right)\right]\right]$

$\Rightarrow u_3(t,x) = \mathcal{L}_t^{-1}\left(2^6\frac{1}{p^5} + 2^8\frac{1}{p^6} + 5\times 2^6\frac{1}{p^7} + 2^7\frac{1}{p^8}\right)$

$\Rightarrow u_3(t,x) = 4\frac{(2t)^4}{4!} + 8\frac{(2t)^5}{5!} + 5\frac{(2t)^6}{6!} + \frac{(2t)^7}{7!}$

$u_4(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\mathcal{L}_t\left(R(u_3)\right)\right]$

$\Rightarrow u_4(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\left[\mathcal{L}_t\left(-\tfrac{32}{315}t^7 - \tfrac{112}{45}t^6 - \tfrac{96}{5}t^5 - \tfrac{160}{3}t^4 - \tfrac{128}{3}t^3\right)\right]\right]$

$\Rightarrow u_4(t,x) = \mathcal{L}_t^{-1}\left[\frac{1}{p^2}\left[\mathcal{L}_t\left(-2^8\frac{t^3}{3!} - 5\times 2^8\frac{t^4}{4!} - 9\times 2^8\frac{t^5}{5!} - 7\times 2^8\frac{t^6}{6!} - 2^9\frac{t^7}{7!}\right)\right]\right]$

$\Rightarrow u_4(t,x) = \mathcal{L}_t^{-1}\left(-2^8\frac{1}{p^6} - 5\times 2^8\frac{1}{p^7} - 9\times 2^8\frac{1}{p^8} - 7\times 2^8\frac{1}{p^9} - 2^9\frac{1}{p^{10}}\right)$

$\Rightarrow u_4(t,x) = -8\frac{(2t)^5}{5!} - 20\frac{(2t)^6}{6!} - 18\frac{(2t)^7}{7!} - 7\frac{(2t)^8}{8!} - \frac{(2t)^9}{9!}$

Step by step, we deduce

$$\sum_{n\geq 0} u_n\left(t,x\right) = \ e^{2x} + \sum_{n\geq 0}\frac{(-2t)^k}{n!}$$

Then

$$u(t,x) = \sum_{n\geq 0} u_n(t,x) = e^{2x} + e^{-2t}$$

The exact solution of model is

$$u(t,x) = e^{2x} + e^{-2t}$$

5 Conclusion

Laplace's Adomian numerical method allowed us to solve some linear partial differential equations by modelling the standard telegraph equation. It is therefore a very powerful numerical analysis tool to solve this type of problem, this method accelerates convergence to the solution . Our study was limited to the linear models of telegraph non-homogeneous reaction, a study of these models in non-homogeneous cases would be an important contribution to the understanding of these models.

Competing Interests

Authors have declared that no competing interests exist.

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