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# Some Fıxed Poınt Theorems ın $A_b$-Metrıc Space

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## Abstract

In this paper, we introduce the notion of $A_b$-metric space and to study its basic topological properties. We prove some fixed point theorems under different contraction and expansion type conditions in the setting of $A_b$-metric space and partially ordered $A_b$-metric space.Our results generalize and extend various results in the existing literature.

*Keywords: $A_b$-metric space; contractive mapping; expansive mapping; fixed point.*

## 1 Introduction

Most of the generalizations for metric fixed point theorems usually start from Banach contraction principle [1]. It is not easy to point out all the generalizations of this principle. The study of expansive mappings is very

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interesting research area of fixed point theory. In 1984, Wang et al. [2] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [3] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. In 1989, Bakhtin [4] introduced the concept of a b-metric space as a generalization of metric spaces, in which many researchers treated the fixed point theory. In 1993, Czerwik [5-6] extended many results related to the b-metric spaces. In 1994, Matthews [7] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 2013, Shukla [8] generalized both the concept of b-metric and partial metric spaces by introducing the partial b-metric spaces. Gähler [9] claimed that 2-metric space is a generalization of an ordinary metric space. He mentioned in [9] that $d(x,y,z)$ geometrically represents the area of a triangle formed by the points $x,y,z \in X$ as its vertices. On the other hand, Ha et al. [10] and Sharma [11] found some mathematical flaws in these claims. It was demonstrated in [11] that $d(x,y,z)$ does not always represent the area of a triangle formed by the points $x,y,z \in X$. Ha et al. [10] proved that the 2-metric is not sequentially continuous in each of its arguments whereas an ordinary metric satisfies this property.

In order to carry out meaningful studies of fixed point results, Dhage [13] suggested an improvement in the basic structure of 2-metric space. In 1984, Dhage in his Ph.D. thesis [13] identified condition second as a weakness in Gähler's theory of a 2-metric space. To overcome these problems, he then introduced the concept of a *D*-metric space. Dhage [14] then studied topological properties of *D*-metric space in a series of papers. Naidu et al. [15] proved that the concepts of convergent sequences and sequential continuity are not well defined in *D*-metric spaces. Naidu et al. [16] pointed out some drawbacks in the idea of open balls in *D*-metric space. In 2003, Mustafa and Sims [17] identified condition third as a weakness in Dhage's theory of *D*-metric space. The tetrahedral inequality in *D*-metric has been replaced with the prototypical rectangular inequality adopted by Mustafa and Sims [18] in 2006 and introduced the notion of *G*-metric space and suggested an important generalization of metric space. Aghajani et al. [19] introduced $G_b$-metric space and established common fixed point of generalized weak contractive mapping in partially ordered $G_b$-metric spaces.

Sedghi et al. [20] have introduced $D^*$-metric spaces which is a probable modification of the definition of *D*-metric spaces introduced by Dhage [13] and proved some basic properties in $D^*$-metric spaces, (see [21]). Every *G*-metric space is a $D^*$-metric space. The converse, however, is false in general; a $D^*$-metric space is not necessarily a *G*-metric space. Sedghi et al. [22] identified condition third of the *G*-metric space as a peculiar limitation but classified the symmetry condition as a common weakness of both *G*- and $D^*$-metric spaces. To overcome these difficulties, Sedghi et al. [22] introduced a new generalized metric space called an *S*-metric space.The S-metric space is a space with three dimensions. Sedghi et al. [22] asserted that every G-metric is an S-metric, see [22, Remarks 1.3] and [22, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [12] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct. Souayah et al. [23] have introduced $S_b$-metric space and established some fixed point theorems.Very recently, Abbas et al. [24] introduced the notion of A-metric space, which generalization of the *S*-metric space.

In this paper, we introduce the notion of $A_b$-metric space and to study its basic topological properties. We also prove some fixed point theorems under different contraction and expansion type conditions in the setting of $A_b$-metric space and partially ordered $A_b$-metric space. Some examples are presented to support the results proved herein. Our results generalize and extend various results in the existing literature.

## 2 Preliminaries

In 2015, Abbas et al. [24] introduced the notion of $A$-metric space.

**Definition 2.1** (see [24]) Let $X$ be a nonempty set. A mapping $A: X^n \to [0,+\infty)$ is called an $A$-metric on $X$ if and only if for all $x_i, a \in X, i = 1,2,3,..n$: the following conditions hold:

(A1). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$,
(A2). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$,
(A3). $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \leq A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a)$
$+A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a)$
$+A(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \cdots$
$+A(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a)$
$+A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)]$.

The pair $(X, A)$ is called an $A$-metric space.

The following is the intuitive geometric example for $A$-metric spaces.

**Example 2.2** (see [24]) Let $X = [1, +\infty)$ . Define $A: X^n \to [0, +\infty)$ by

$$A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|$$

for all $x_i \in X, i = 1,2, \dots n$.

**Example 2.3** (see [24]) Let $= \mathbb{R}$ . Define $A_b: X^n \to [0, +\infty)$ by

$$\begin{aligned} A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = & \left|\textstyle\sum_{i=n}^{2} x_i - (n-1)x_1\right| \\ & + \left|\textstyle\sum_{i=n}^{3} x_i - (n-2)x_2\right| + \cdots \\ & + \left|\textstyle\sum_{i=n}^{n-3} x_i - 3x_{n-3}\right| \\ & + \left|\textstyle\sum_{i=n}^{n-2} x_i - 2x_{n-2}\right| \\ & + |x_n - x_{n-1}| \end{aligned}$$

for all $x_i \in X, i = 1,2, \dots n$.

**Lemma 2.4** (see [24]) Let $(X, A)$ be an $A$-metric space. Then for all $x, y \in X$,

$$A_b(x, x, x, x, \dots, (x)_{n-1}, y) = A_b(y, y, y, y \dots, (y)_{n-1}, x)$$

**Lemma 2.5** (see [24]) Let $(X, A)$ be an $A$-metric space. Then for all $x, y, z \in X$,

$$\begin{aligned} A_b(x, x, x, x, \dots, (x)_{n-1}, z) \leq & (n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\ & + A_b(z, z, z, z, \dots, (z)_{n-1}, y) \end{aligned}$$

and

$$\begin{aligned} A_b(x, x, x, x, \dots, (x)_{n-1}, z) \leq & (n-1)A_b(x, x, x, x, \dots, (x)_{n-1}, y) \\ & + A_b(y, y, y, y, \dots, (y)_{n-1}, z) \end{aligned}$$

**Lemma 2.6** (see [24]) Let $(X, A)$ be an $A$-metric space. Then $(X \times X, D_A)$ is an $A$-metric space on $X \times X$, where $D_A$ is given by for all $x_i, y_j \in X, i, j = 1,2, \dots, n$:

$$D_A\big((x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)\big) = A(x_1, x_2, x_3, \dots, x_n) + A(y_1, y_2, y_3, \dots, y_n).$$

**Definition 2.7** (see [24]) Let $(X, A)$ be an $A$-metric space. Then

1. A sequence $\{x_k\}$ is called convergent to $x$ in $(X, A)$ if

$$\lim_{k\to+\infty} A(x_k, x_k, x_k, x_k, \dots (x_k)_{n-1}, x) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have

$$A(x_k, x_k, x_k, x_k, ..(x_k)_{n-1}, x) \leq \epsilon$$

and we write $\lim_{k\to+\infty} x_k = x$.

2. A sequence $\{x_k\}$ is called Cauchy in $(X, A)$ if

$$\lim_{k,m\to+\infty} A(x_k, x_k, x_k, x_k, ..(x_k)_{n-1}, x_m) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \geq n_0$, we have

$$A(x_k, x_k, x_k, x_k, ..(x_k)_{n-1}, x_m) \leq \epsilon.$$

3. $(X, A)$ is said to be complete if every Cauchy sequence in $(X, A)$ is a convergent.

**Lemma 2.8** (see [24]) Let $(X, A)$ be an $A$-metric space. If the sequence $\{x_k\}$ in $X$ converges to $x$, then $x$ is unique.

**Lemma 2.9** (see [24]) Every convergent sequence in $A$-metric space $(X, A)$ is a Cauchy sequence.

# 3 $A_b$-Metric Space

We now present the concept of an $A_b$-metric space and study some of its properties needed in the sequel.

**Definition 3.1** Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $A_b: X^n \to [0,+\infty)$ is called an $A_b$-metric on $X$ if and only if for all $x_i, a \in X, i = 1,2,3, .. n$: the following conditions hold:

(Ab1). $A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$;
(Ab2). $A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$;
(Ab3).
$$\begin{aligned} A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \leq s[&A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\ &+A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ &+A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \cdots \\ &+A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\ &+A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)]. \end{aligned}$$

The pair $(X, A_b)$ is called an$A_b$-metric space.

Note that the class of $A_b$-metric spaces is larger than the class of $A$-metric spaces. Indeed, every $A$-metric space is an $A_b$-metric space with $s = 1$. However, the converse is not always true. Also $A_b$-metric space is an $n$-dimensional $S_b$-metric space. Therefore the $S_b$-metric are special cases of an $A_b$-metric with $n = 3$.

The following is the intuitive geometric example for $A_b$-metric spaces.

**Example 3.2** Let = $[1,+\infty)$ . Define $A_b: X^n \to [0,+\infty)$ by

$$A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} \left|x_i - x_j\right|^2$$

for all $x_i \in X, i = 1,2, \dots n$.

**Proof:** Obviously (Ab1) and (Ab2) are satisfied. We shall show that, for all$x_i, a \in X, i = 1,2, \dots n$, (Ab3) is valid. Note that

$$
\begin{aligned}
A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) &= (n-1)|x_1 - a|^2,\\
A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) &= (n-1)|x_2 - a|^2,\\
A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) &= (n-1)|x_3 - a|^2\\
&\vdots\\
A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) &= (n-1)|x_{n-1} - a|^2.\\
A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a) &= (n-1)|x_n - a|^2
\end{aligned}
$$

for all $x_i, a \in X, i = 1,2, \dots n$.

Now

$$
\begin{aligned}
A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &= \sum_{i=1}^{n} \sum_{i<j} \left|x_i - x_j\right|^2\\
&= \sum_{i=1}^{n} \sum_{i<j} \left|(x_i - a) - (x_j - a)\right|^2\\
&\leq \{2(n-1)|x_1 - a|^2 + 2|x_2 - a|^2 + 2|x_3 - a|^2 + \cdots\\
&+2|x_n - a|^2\} + \{2(n-2)|x_2 - a|^2 + 2|x_3 - a|^2 + \cdots\\
&+2|x_n - a|^2\} + \{2(n-3)|x_3 - a|^2 + 2|x_4 - a|^2 + \cdots\\
&+2|x_n - a|^2\} + \cdots + \{2(2)|x_{n-2} - a|^2 + 2|x_{n-1} - a|^2\\
&+2|x_n - a|^2\} + 2|x_{n-1} - a|^2 + 2|x_n - a|^2\\
&= 2(n-1)|x_1 - a|^2 + 2(n-1)|x_2 - a|^2 + \cdots\\
&+2(n-1)|x_{n-1} - a|^2 + 2(n-1)|x_n - a|^2\\
&= 2[A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a)\\
&+A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a)\\
&+A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) + \cdots\\
&+A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a)\\
&+A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a)].
\end{aligned}
$$

for all $x_i, a \in X, i = 1,2, \dots n$. Therefore, $(X, A_b)$ is an $A_b$-metric space with $s = 2 > 1$.

**Example 3.3** Let $= \mathbb{R}$. Define $A_b: X^n \to [0,+\infty)$by

$$
\begin{aligned}
A_b(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &= \left|\sum_{i=n}^{2} x_i - (n-1)x_1\right|^2\\
&+\left|\sum_{i=n}^{3} x_i - (n-2)x_2\right|^2 + \cdots\\
&+\left|\sum_{i=n}^{n-3} x_i - 3x_{n-3}\right|^2\\
&+\left|\sum_{i=n}^{n-2} x_i - 2x_{n-2}\right|^2\\
&+|x_n - x_{n-1}|^2
\end{aligned}
$$

for all $x_i \in X, i = 1,2, \dots n$.

**Proof:** Clearly conditions (Ab1) and (Ab2) are satisfied.We shall show that, for all $x_i, a \in X, i = 1,2, \dots n$, (Ab3) is valid. Note that

$$
\begin{aligned}
A_b(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) &= (n-1)|a - x_1|^2,\\
A_b(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) &= (n-1)|a - x_2|^2,\\
A_b(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) &= (n-1)|a - x_3|^2\\
&\vdots\\
A_b(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) &= (n-1)|a - x_{n-1}|^2.\\
A_b(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a) &= (n-1)|a - x_n|^2
\end{aligned}
$$

for all$x_i, a \in X, i = 1,2, \dots n$.

Now

$$\begin{aligned}A_b(x_1,x_2,x_3,\dots,x_{n-1},x_n) &= |\textstyle\sum_{i=n}^{2}(x_i-a)-(n-1)(x_1-a)|^2\\
&+\left|\textstyle\sum_{i=n}^{3}(x_i-a)-(n-2)(x_2-a)\right|^2+\cdots\\
&+\left|\textstyle\sum_{i=n}^{n-2}(x_i-a)-3(x_{n-3}-a)\right|^2\\
&+|\textstyle\sum_{i=n}^{n-1}(x_i-a)-2(x_{n-2}-a)|^2\\
&+|(x_n-a)-(x_{n-1}-a)|^2\\
&\le 2\textstyle\sum_{i=n}^{2}|x_i-a|^2+2(n-1)|x_1-a|^2\\
&+2\textstyle\sum_{i=n}^{3}|x_i-a|^2+2(n-2)|x_2-a|^2+\cdots\\
&+2\textstyle\sum_{i=n}^{n-2}|x_i-a|^2+2(3)|x_{n-3}-a|^2\\
&+2\textstyle\sum_{i=n}^{n-1}|x_i-a|^2+2(2)|x_{n-2}-a|^2\\
&+2|x_n-a|^2+2|x_{n-1}-a|^2\\
&=2|x_n-a|^2+2|x_{n-1}-a|^2+\cdots+2|x_2-a|^2\\
&+2(n-1)|x_1-a|^2+2|x_n-a|^2+2|x_{n-1}-a|^2\\
&+\cdots+2|x_3-a|^2+2(n-2)|x_2-a|^2+\cdots\\
&+2|x_n-a|^2+2|x_{n-1}-a|^2+2(3)|x_{n-3}-a|^2\\
&+|x_n-a|^2+|x_{n-1}-a|^2+2(2)|x_{n-2}-a|^2\\
&+2|x_n-a|^2+2|x_{n-1}-a|^2\\
&=2(n-1)|x_n-a|^2+2(n-1)|x_{n-1}-a|^2\\
&+2(n-1)|x_{n-2}-a|^2+\cdots\\
&+2(n-1)|x_3-a|^2+2(n-1)|x_2-a|^2\\
&+2(n-1)|x_1-a|^2\\
&=2(n-1)\{|x_1-a|^2+|x_2-a|^2+\cdots+|x_n-a|^2\}\\
&=2[A_b(x_1,x_1,x_1,\dots,(x_1)_{n-1},a)\\
&+A_b(x_2,x_2,x_2,\dots,(x_2)_{n-1},a)\\
&+A_b(x_3,x_3,x_3,\dots,(x_3)_{n-1},a)+\cdots\\
&+A_b(x_{n-1},x_{n-1},x_{n-1},\dots,(x_{n-1})_{n-1},a)\\
&+A_b(x_n,x_n,x_n,\dots,(x_n)_{n-1},a)].\end{aligned}$$

for all $x_i, a \in X, i = 1,2,\dots n$. Therefore, $(X, A_b)$ is an $A_b$-metric space with $s = 2 > 1$.

**Lemma 3.4** Let $(X, A_b)$ be an $A_b$-metric space with $s \ge 1$. Then for all $x, y \in X$,

$$A_b(x,x,x,x,\dots,(x)_{n-1},y) \le sA_b(y,y,y,y\dots,(y)_{n-1},x)$$

**Proof** For all $x, y \in X$, applying (Ab3), we have

$$\begin{aligned}A_b(x,x,x,\dots,(x)_{n-1},y) &\le s[A_b(x,x,x,x\dots,(x)_{n-1},x)\\
&+A_b(x,x,x,x\dots,(x)_{n-1},x)\\
&+A_b(x,x,x,x\dots,(x)_{n-1},x)+\cdots\\
&+\big(A_b(x,x,x,x\dots,(x)_{n-1},x)\big)_{n-1}\\
&+A_b(y,y,y,y\dots,(y)_{n-1},x)]\\
&=sA_b(y,y,y,y\dots,(y)_{n-1},x)\end{aligned}$$

**Lemma 3.5** Let $(X, A_b)$ be an $A_b$-metric space with $s \ge 1$. Then for all $x, y, z \in X$,

$$\begin{aligned}A_b(x,x,x,x,\dots,(x)_{n-1},z) &\le s[(n-1)A_b(x,x,x,x,\dots,(x)_{n-1},y)\\
&+A_b(z,z,z,z,\dots,(z)_{n-1},y)]\end{aligned}$$

and

$$\begin{aligned}A_b(x,x,x,x,\dots,(x)_{n-1},z) &\le s[(n-1)A_b(x,x,x,x,\dots,(x)_{n-1},y)\\
&+sA_b(y,y,y,y,\dots,(y)_{n-1},z)]\end{aligned}$$

**Proof** For all $x, y, z \in X$, applying (Ab3), we have

$$\begin{aligned}A_b(x,x,x,\dots,(x)_{n-1},z) &\le s[A_b(x,x,x,x\dots.,(x)_{n-1},y)\\&+A_b(x,x,x,x\dots.,(x)_{n-1},y)\\&+A_b(x,x,x,x\dots.,(x)_{n-1},y)+\cdots\\&+\left(A_b(x,x,x,x\dots.,(x)_{n-1},y)\right)_{n-1}\\&+A_b(z,z,z,z\dots.,(z)_{n-1},y)]\\&=s[(n-1)A_b(x,x,x,x,\dots,(x)_{n-1},y)\\&+A_b(z,z,z,z,\dots,(z)_{n-1},y)]\end{aligned}$$

which implies that

$$\begin{aligned}A_b(x,x,x,x,\dots,(x)_{n-1},z) \le s[&(n-1)A_b(x,x,x,x,\dots,(x)_{n-1},y)\\&+A_b(z,z,z,z,\dots,(z)_{n-1},y)]\end{aligned}$$

Using Lemma 3.4, we have

$$\begin{aligned}A_b(x,x,x,x,\dots,(x)_{n-1},z) \le s[&(n-1)A_b(x,x,x,x,\dots,(x)_{n-1},y)\\&+sA_b(y,y,y,y,\dots..,(y)_{n-1},z)].\end{aligned}$$

**Lemma 3.6** Let $(X,A_b)$ be an $A_b$-metric space with $s\ge 1$. Then $(X\times X,D_{A_b})$ is an $A_b$-metric space with $s\ge 1$ on $X\times X$, where $D_{A_b}$ is given by for all $x_i,y_j\in X, i,j=1,2,\dots,n$:

$$D_{A_b}\left((x_1,y_1),(x_2,y_2),(x_3,y_3),\dots,(x_n,y_n)\right)=A_b(x_1,x_2,x_3,\dots.,x_n)+A_b(y_1,y_2,y_3,\dots.,y_n).$$

**Proof** For all $x_i,y_j\in X, i,j=1,2,\dots,n$; we have

$$\begin{aligned}&A_b(x_1,x_2,x_3,\dots.,x_n)+A_b(y_1,y_2,y_3,\dots.,y_n)\ge 0\\&\quad\Rightarrow\ D_{A_b}\left((x_1,y_1),(x_2,y_2),(x_3,y_3),\dots,(x_n,y_n)\right)\ge 0.\end{aligned}$$

Also

$$\begin{aligned}&D_{A_b}\left((x_1,y_1),(x_2,y_2),(x_3,y_3),\dots,(x_n,y_n)\right)=0\\&\quad\Leftrightarrow\ A_b(x_1,x_2,x_3,\dots.,x_n)+A_b(y_1,y_2,y_3,\dots.,y_n)=0\\&\quad\Leftrightarrow\ A_b(x_1,x_2,x_3,\dots.,x_n)=0 \text{ and } A_b(y_1,y_2,y_3,\dots.,y_n)=0\\&\quad\Leftrightarrow\ x_1=x_2=x_3=\cdots=x_n \text{ and } y_1=y_2=y_3=\cdots=y_n\\&\quad\Leftrightarrow (x_1,y_1)=(x_2,y_2)=(x_3,y_3)=\cdots=(x_n,y_n).\end{aligned}$$

Consider

$$\begin{aligned}&D_{A_b}\left((x_1,y_1),(x_2,y_2),(x_3,y_3),\dots,(x_n,y_n)\right)\\&\quad=A_b(x_1,x_2,x_3,\dots.,x_n)+A_b(y_1,y_2,y_3,\dots.,y_n)\\&\quad\le s[A_b(x_1,x_1,x_1,\dots.,(x_1)_{n-1},a)+A_b(x_2,x_2,x_2,\dots.,(x_2)_{n-1},a)\\&\quad+A_b(x_3,x_3,x_3,\dots.,(x_3)_{n-1},a)+\cdots\\&\quad+A_b(x_{n-1},x_{n-1},x_{n-1},\dots.,(x_{n-1})_{n-1},a)+A_b(x_n,x_n,x_n,\dots.,(x_n)_{n-1},a)]\\&\quad+s[A_b(y_1,y_1,y_1,\dots.,(y_1)_{n-1},b)+A_b(y_2,y_2,y_2,\dots.,(y_2)_{n-1},b)\\&\quad+A_b(y_3,y_3,y_3,\dots.,(y_3)_{n-1},b)+\cdots\\&\quad+A_b(y_{n-1},y_{n-1},y_{n-1},\dots.,(y_{n-1})_{n-1},b)+A_b(y_n,y_n,y_n,\dots.,(y_n)_{n-1},b)]\\&\quad=s[A_b(x_1,x_1,x_1,\dots.,(x_1)_{n-1},a)+A_b(y_1,y_1,y_1,\dots.,(y_1)_{n-1},b)]\\&\quad+s[A_b(x_2,x_2,x_2,\dots.,(x_2)_{n-1},a)+A_b(y_2,y_2,y_2,\dots.,(y_2)_{n-1},b)]\\&\quad+s[A_b(x_3,x_3,x_3,\dots.,(x_3)_{n-1},a)+A_b(y_3,y_3,y_3,\dots.,(y_3)_{n-1},b)]+\cdots\\&\quad+s[A_b(x_{n-1},x_{n-1},,\dots.,(x_{n-1})_{n-1},a)+A_b(y_{n-1},y_{n-1},,\dots.,(y_{n-1})_{n-1},b)]\\&\quad+s[A_b(x_n,x_n,x_n,\dots.,(x_n)_{n-1},a)+A_b(y_n,y_n,y_n,\dots.,(y_n)_{n-1},b)]\\&\quad=s\left[D_{A_b}\left((x_1,y_1),(x_1,y_1),(x_1,y_1),\dots,(a,b)\right)\right.\end{aligned}$$

$$+D_{A_b}\big((x_2,y_2),(x_2,y_2),(x_2,y_2),\dots,(a,b)\big)+\cdots$$
$$+D_{A_b}\big((x_n,y_n),(x_n,y_n),(x_n,y_n),\dots,(a,b)\big)\big]$$

Thus,

$$D_{A_b}\big((x_1,y_1),(x_2,y_2),(x_3,y_3),\dots,(x_n,y_n)\big)\le s\big[D_{A_b}\big((x_1,y_1),(x_1,y_1),(x_1,y_1),\dots,(a,b)\big)$$
$$+D_{A_b}\big((x_2,y_2),(x_2,y_2),(x_2,y_2),\dots,(a,b)\big)+\cdots$$
$$+D_{A_b}\big((x_n,y_n),(x_n,y_n),(x_n,y_n),\dots,(a,b)\big)\big]$$

Hence $(X\times X, D_{A_b})$ is an $A_b$-metric space with $s\ge 1$ on $X\times X$.

**Remark 3.7**

(a). If we put $s=1$, then we have

$$D_A\big((x_1,y_1),(x_2,y_2),(x_3,y_3),\dots,(x_n,y_n)\big)=A(x_1,x_2,x_3,\dots.,x_n)+A(y_1,y_2,y_3,\dots.,y_n).$$

Then $(X\times X, D_A)$ is an $A$-metric space on $X\times X$.

(b). If we put $n=3$, then we have

$$D_{A_b}\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big)=S_b(x_1,x_2,x_3)+S_b(y_1,y_2,y_3)$$

Then $(X\times X, D_{A_b})$ is an $S_b$-metric space on $X\times X$.

(c). If we put $n=3, s=1$, then we have

$$D_A\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big)=S(x_1,x_2,x_3)+S(y_1,y_2,y_3)$$

Then $(X\times X, D_A)$ is an $S$-metric space on $X\times X$.

Note also that the following implications hold.

| | | | | | | |
|---|---|---|---|---|---|---|
| $G$-metric space | $\Rightarrow$ | $D^*$-metric space | $\Rightarrow$ | $S$-metric space | $\Rightarrow$ | $A$-metric space |
| $\Downarrow$ | | | | $\Downarrow$ | | $\Downarrow$ |
| $G_b$-metric space | | $\Rightarrow$ | | $S_b$-metric space | $\Rightarrow$ | $A_b$-metric space |

**Definition 3.8** Let $(X, A_b, s)$ be an$A_b$-metric space. Then

(1). A sequence $\{x_k\}$ is called convergent to $x$ in $(X, A_b)$ if $\lim_{k\to+\infty} A_b(x_k,x_k,x_k,x_k,\dots,(x_k)_{n-1},x)=0$ .That is, for each $\epsilon\ge 0$, there exists $n_0\in$ such that for all $k\ge n_0$, we have $A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},x)\le\epsilon$ and we write $\lim_{k\to+\infty} x_k=x$.

(2). A sequence $\{x_k\}$ is called Cauchy in $(X, A_b)$ if $\lim_{k,m\to+\infty} A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},x_m)=0$. That is, for each $\epsilon\ge 0$, there exists $n_0\in\mathbb{N}$ such that for all $k,m\ge n_0$, we have $A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},x_m)\le\epsilon$.

(3). $(X, A_b)$ is said to be complete if every Cauchy sequence in $(X, A_b)$is a convergent.

**Lemma 3.9** Let $(X, A_b)$ be an $A_b$-metric space with $s\ge 1$. If the sequence $\{x_k\}$ in $X$ converges to $x$, then $x$ is unique.

**Proof** On the contrary, assume that $\{x_k\}$ converges to $x$ and $y$. Then given $\epsilon > 0$,there exist $n_1, n_2 \in \mathbb{N}$ such that for all $k \geq n_1$, we have

$$A_b(x_k, x_k, x_k, x_k, ..(x_k)_{n-1}, x) < \frac{\epsilon}{2s^2(n-1)}$$

and for all $k \geq n_2$, we have

$$A_b(x_k, x_k, x_k, x_k, ..(x_k)_{n-1}, y) < \frac{\epsilon}{2s^2}.$$

Choose $n_0 = \max\{n_1, n_2\}$, therefore for all $k \geq n_0$, we have

$$\begin{aligned} A_b(x, x, x, x \dots, (x)_{n-1}, y) &\leq s[A_b(x, x, x, x \dots., (x)_{n-1}, x_k) \\ &+A_b(x, x, x, x \dots., (x)_{n-1}, x_k) \\ &+A_b(x, x, x, x \dots., (x)_{n-1}, x_k) + \cdots \\ &+\left(A_b(x, x, x, x \dots., (x)_{n-1}, x_k)\right)_{n-1} \\ &+A_b(y, y, y, y, \dots., (y)_{n-1}, x_k)] \\ &= s(n-1)A_b(x, x, x, x \dots., (x)_{n-1}, x_k) \\ &+sA_b(y, y, y, y, \dots., (y)_{n-1}, x_k) \end{aligned}$$

Hence from Lemma 3.4, we have

$$\begin{aligned} A_b(x, x, x, x \dots, (x)_{n-1}, y) &\leq s^2(n-1)A_b(x_k, x_k, x_k, x_k, \dots., (x_k)_{n-1}, x) \\ &+s^2 A_b(x_k, x_k, x_k, x_k, \dots., (x_k)_{n-1}, y) \\ &< s^2(n-1) \times \frac{\epsilon}{2s^2(n-1)} + s^2 \times \frac{\epsilon}{2s^2} \\ &= \epsilon. \end{aligned}$$

Since $\epsilon$ is arbitrary, we have $A_b(x, x, x, x \dots, (x)_{n-1}, y) = 0$ and so $x = y$. Establishing the uniqueness of $\{x_k\}$.

**Lemma 3.10** Every convergent sequence in $A_b$-metric space $(X, A_b)$ is a Cauchy sequence.

**Proof** Let $\{x_k\}$ be convergent in$(X, A_b)$. Let $\lim_{k\to+\infty} x_k = x$. Then given $\epsilon > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that for all $k \geq n_1$, we have

$$A_b(x_k, x_k, x_k, x_k, ..(x_k)_{n-1}, x) < \frac{\epsilon}{2s(n-1)}$$

and for all $m \geq n_2$, we have

$$A_b(x_m, x_m, x_m, x_m, ..(x_m)_{n-1}, x) < \frac{\epsilon}{2s}.$$

Choose $n_0 = \max\{n_1, n_2\}$, therefore for all $k, m \geq n_0$, we have

$$\begin{aligned} A_b(x_k, x_k, x_k, x_k, ..(x_k)_{n-1}, x_m) &\leq s(n-1)A_b(x_k, x_k, x_k, x_k, ..(x_k)_{n-1}, x) \\ &+sA_b(x_m, x_m, x_m, x_m, \dots., (x_m)_{n-1}, x) \\ &< s(n-1) \times \frac{\epsilon}{2s(n-1)} + s \times \frac{\epsilon}{2s} \\ &= \epsilon. \end{aligned}$$

This implies that $\{x_k\}$ is a Cauchy sequence.

**Lemma 3.11** Let $(X, A_b)$ be an $A_b$-metric space with $s \geq 1$. If $\lim_{k\to+\infty} x_k = x$ and $\lim_{k\to+\infty} y_k = y$, then

$$\begin{aligned}\frac{1}{s^2}A_b(x,x,x,x,..(x)_{n-1},y) &\le \lim_{k\to+\infty}\inf A_b(x_k,x_k,x_k,x_k,\dots(x_k)_{n-1},y_k)\\ &\le \lim_{k\to+\infty}\sup A_b(x_k,x_k,x_k,x_k,\dots(x_k)_{n-1},y_k)\\ &\le s^2A_b(x,x,x,x,..(x)_{n-1},y)\end{aligned}$$

In particular, if $y_k = y$ is constant, then

$$\begin{aligned}\frac{1}{s^2}A_b(x,x,x,x,..(x)_{n-1},y) &\le \lim_{k\to+\infty}\inf A_b(x_k,x_k,x_k,x_k,\dots(x_k)_{n-1},y)\\ &\le \lim_{k\to+\infty}\sup A_b(x_k,x_k,x_k,x_k,\dots(x_k)_{n-1},y)\\ &\le s^2A_b(x,x,x,x,..(x)_{n-1},y)\end{aligned}$$

**Proof** Using (Ab3), we have

$$\begin{aligned}A_b(x,x,x,x,..(x)_{n-1},y) &\le s(n-1)A_b(x,x,x,x,..(x)_{n-1},x_k)\\ &\quad +sA_b(y,y,y,y,..(y)_{n-1},x_k)\\ &\le s(n-1)A_b(x,x,x,x,..(x)_{n-1},x_k)\\ &\quad +s^2(n-1)A_b(y,y,y,y,..(y)_{n-1},y_k)\\ &\quad +s^2A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},y_k)\\ &\le s^2(n-1)A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},x)\\ &\quad +s^3(n-1)A_b(y_k,y_k,y_k,y_k,..(y_k)_{n-1},y)\\ &\quad +s^2A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},y_k)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},y_k) &\le s(n-1)A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},x)\\ &\quad +sA_b(y_k,y_k,y_k,y_k,..(y_k)_{n-1},x)\\ &\le s(n-1)A_b(x_k,x_k,x_k,x_k,..(x_k)_{n-1},x)\\ &\quad +s^2(n-1)A_b(y_k,y_k,y_k,y_k,..(y_k)_{n-1},y)\\ &\quad +s^2A_b(x,x,x,x,..(x)_{n-1},y)\end{aligned}$$

Taking the lower limit as $k\to+\infty$, in the first inequality and the upper limit as $k\to+\infty$ in the second inequality we obtain

$$\begin{aligned}\frac{1}{s^2}A_b(x,x,x,x,..(x)_{n-1},y) &\le \lim_{k\to+\infty}\inf A_b(x_k,x_k,x_k,x_k,\dots(x_k)_{n-1},y_k)\\ &\le \lim_{k\to+\infty}\sup A_b(x_k,x_k,x_k,x_k,\dots(x_k)_{n-1},y_k)\\ &\le s^2A_b(x,x,x,x,..(x)_{n-1},y)\end{aligned}$$

If $y_k = y$, then using (Ab3), we have

$$\begin{aligned}A_b(x,x,x,x,\dots,(x)_{n-1},y) &\le s(n-1)A_b(x,x,x,x,\dots,(x)_{n-1},x_k)\\ &\quad +sA_b(y,y,y,y,\dots,(y)_{n-1},x_k)\\ &\le s^2(n-1)A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x)\\ &\quad +s^2A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},y)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}A_b(x_k,x_k,x_k,x_k,\dots,(x_k)_{n-1},y) &\le s(n-1)A_b(x_k,x_k,x_k,x_k,\dots,(x_k)_{n-1},x)\\ &\quad +sA_b(y,y,y,y,\dots,(y)_{n-1},x)\\ &\le s(n-1)A_b(x_k,x_k,x_k,x_k,\dots,(x_k)_{n-1},x)\\ &\quad +s^2A_b(x,x,x,\dots,(x)_{n-1},y)\end{aligned}$$

Taking the lower limit as $k\to+\infty$, in the first inequality and the upper limit as $k\to+\infty$ in the second inequality we obtain

$$\frac{1}{s^2}A_b(x,x,x,x,..(x)_{n-1},y) \le \lim_{k\to+\infty}\inf A_b(x_k,x_k,x_k,x_k,\dots(x_k)_{n-1},y) \le \lim_{k\to+\infty}\sup A_b(x_k,x_k,x_k,x_k,\dots(x_k)_{n-1},y) \le s^2 A_b(x,x,x,x,..(x)_{n-1},y)$$

**Definition 3.12** The $A_b$-metric space $(X, A_b)$ is said to be bounded if there exists a constant $r > 0$ such that $A_b(x,x,x,\dots,x,y) \le r$ for all $x, y \in X$. Otherwise, $X$ is unbounded.

**Definition 3.13** Given a point $x_0$ in $A_b$-metric space $(X, A_b)$ and a positive real number $r$, the set $B(x_0,r) = \{y \in X: A_b(y,y,y,\dots,y,x_0) < r\}$ is called an open ball centered at $x_0$ with radius $r$.

The set $\overline{B(x_0,r)} = \{y \in X: A_b(y,y,y,\dots,y,x_0) \le r\}$ is called a closed ball centered at $x_0$ with radius $r$.

**Definition 3.14** A subset $G$in $A_b$-metric space $(X, A_b)$ is said to be an open set if for each $x \in G$ there exists an $r > 0$ such that $B(x,r) \subset G$. A subset $F \subset X$ is called closed if $X \setminus F$ is open.

**Definition 3.15** Let $(X, A_b)$ be an $A_b$-metric spacewith $s \ge 1$. A map $f: X \to X$ is said to be contraction if there exits a constant $\lambda \in [0,1)$ such that

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) \le \lambda A_b(x^1, x^2, x^3, \dots, x^n)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$. In case

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) < A_b(x^1, x^2, x^3, \dots, x^n)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$, $f$ is called contractive mapping.

**Definition 3.16** Let $(X, A_b)$ be an $A_b$-metric spacewith $s \ge 1$. A map $f: X \to X$ is said to be expansion mapping if there exits $\lambda > 1$ such that

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) \ge \lambda A_b(x^1, x^2, x^3, \dots, x^n)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$. In case

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) > A_b(x^1, x^2, x^3, \dots, x^n)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$, $f$ is called expansive mapping.

# 4 Fixed Point Theorems for Contraction Mapping

We begin with a simple but a useful lemma.

**Lemma 4.1** Let $(X, A_b)$ be an $A_b$-metric space with $s \ge 1$ and $\{x_k\}$ be a sequence in $(X, A_b)$ such that

$$A_b(x_k,x_k,x_k,..(x_k)_{n-1},x_{k+1}) \le \lambda A_b(x_{k-1},x_{k-1},x_{k-1},..(x_{k-1})_{n-1},x_k) \quad (1)$$

where $\lambda \in \left[0, \frac{1}{s^2}\right)$ and $k = 1, 2, \dots$. Then $\{x_k\}$ is a Cauchy sequence in $(X, A_b)$.

**Proof** For $k = 1, 2, ..$, we get by induction

$$
\begin{aligned}
A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_{k+1}) &\le \lambda A_b(x_{k-1}, x_{k-1}, x_{k-1}, ..(x_{k-1})_{n-1}, x_k) \\
&\le \lambda^2 A_b(x_{k-2}, x_{k-2}, x_{k-2}, ..(x_{k-2})_{n-1}, x_{k-1}) \\
&\vdots \\
&\le \lambda^k A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) \qquad (2)
\end{aligned}
$$

Let $m > k$. It follows that

$$
\begin{aligned}
A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_m) &\le s[(n-1)A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_{k+1}) \\
&+A_b(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+1})] \\
&\le s(n-1)A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_{k+1}) \\
&+s^2 A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_m) \\
&\le s(n-1)A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_{k+1}) \\
&+s^3[(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2}) \\
&+A_b(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+2})] \\
&\le s(n-1)A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_{k+1}) \\
&+s^3(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2}) \\
&+s^4 A_b(x_{k+2}, x_{k+2}, x_{k+2}, \dots, (x_{k+2})_{n-1}, x_m)] \\
&\le s(n-1)A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_{k+1}) \\
&+s^3(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2}) \\
&+s^5[(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+3}) \\
&+A_b(x_m, x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+3})] \\
&\le s(n-1)A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_{k+1}) \\
&+s^3(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+2}) \\
&+s^5(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+3}) \\
&+s^7(n-1)A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{k+4}) + \cdots \\
&+s^{2m-2k-3}(n-1)A_b(x_{m-2}, x_{m-2}, x_{m-2}, \dots, (x_{m-2})_{n-1}, x_{m-1}) \\
&+s^{2m-2k-2}A_b(x_{m-1}, x_{m-1}, x_{m-1}, \dots, (x_{m-1})_{n-1}, x_m) \\
&\le (n-1)[s\lambda^k + s^3\lambda^{k+1} + s^5\lambda^{k+2} + s^7\lambda^{k+3} + \cdots + s^{2m-2k-3}\lambda^{m-2}] \\
&\times A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) \\
&+s^{2m-2k-2}\lambda^{m-1} \times A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) \\
&= (n-1)s\lambda^k[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots + s^{2m-2k-4}\lambda^{m-k-2}] \\
&\times A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) \\
&+s^{2m-2k-3}\lambda^{m-k-1} \times A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) \\
&\le (n-1)s\lambda^k[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + \cdots] \\
&\times A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) \\
&\le (n-1)\frac{s\lambda^k}{1-\lambda s^2} A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) \qquad (3)
\end{aligned}
$$

Since $\lambda s^2 < 1$. Assume that $A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) > 0$. By taking limit as $k, m \to +\infty$ in above inequality we get

$$\lim_{k,m\to+\infty} A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_m) = 0.$$

Therefore, $\{x_k\}$ is a Cauchy sequence in $X$. Also, if $A_b(x_0, x_0, x_0, x_0, ..(x_0)_{n-1}, x_1) = 0$, then $A_b(x_k, x_k, x_k, ..(x_k)_{n-1}, x_m) = 0$ for all $m > k$ and hence $\{x_k\}$ is a Cauchy sequence in $X$.

**Theorem 4.2** Let $(X, A_b)$ be a complete $A_b$-metric space with $s \ge 1$ and $f: X \to X$ be a continuous mapping satisfy

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) \le \psi[A_b(x^1, x^2, x^3, \dots, x^n)] \qquad (4)$$

for all $x^1, x^2, x^3, \dots, x^n \in X$, where $\psi: [0, +\infty) \to [0, +\infty)$ is an increasing function such that $\lim_{k\to\infty} \psi^k(t) = 0$ for each fixed $t > 0$. Then $f$ has a unique fixed point in $X$.

**Proof** Let $x \in X$ and $\epsilon > 0$. Let $m$ be a natural number such that $\psi^m(\epsilon) < \frac{\epsilon}{2s^2(n-1)}$. Let $F = f^m$ and $x_k = F^k(x)$ for $k \in \mathbb{N}$. Then for all $x, y \in X$ and $\varphi = \psi^m$, we have

$$\begin{aligned} A_b(Fx, Fx, Fx, \dots, (Fx)_{n-1}, Fy) &\le \psi^m\big(A_b(x, x, x, \dots, (x)_{n-1}, y)\big) \\ &= \varphi\big(A_b(x, x, x, \dots, (x)_{n-1}, y)\big) \end{aligned} \tag{5}$$

Hence, for each $k \in \mathbb{N}$, $A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_k) \to 0$ as $k \to \infty$. Therefore, let $k$ be such that

$$A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) < \frac{\epsilon}{2s^2(n-1)} \tag{6}$$

Let's define the ball $B(x_k, \epsilon)$ such that for every $z \in B(x_k, \epsilon) = \{y \in X \mid A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, y)\}$. Note that $x_k \in B(x_k, \epsilon)$, therefore $B(x_k, \epsilon) \neq \emptyset$. Hence, for all $z \in B(x_k, \epsilon)$ we have

$$\begin{aligned} A_b(Fx_k, Fx_k, Fx_k, \dots, (Fx_k)_{n-1}, Fz) &\le \varphi\big(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, z)\big) \\ &\le \varphi(\epsilon) = \psi^m(\epsilon) < \frac{\epsilon}{2s^2(n-1)} < \frac{\epsilon}{2s^2} \end{aligned} \tag{7}$$

Thus,

$$\begin{aligned} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, Fz) &\le s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad + sA_b(Fz, Fz, Fz, \dots, (Fz)_{n-1}, x_{k+1}) \\ &\le s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad + s^2 A_b(x_{k+1}, x_{k+1}, x_{k+1} \dots, (x_{k+1})_{n-1}, Fz) \\ &< s^2(n-1) \times \frac{\epsilon}{2s^2(n-1)} + s^2 \times \frac{\epsilon}{2s^2} = \epsilon. \end{aligned} \tag{8}$$

Hence, $F$ maps $B(x_k, \epsilon)$ to it self. Since $x_k \in B(x_k, \epsilon)$, we have $Fx_k \in B(x_k, \epsilon)$. By repeating this process we get

$$F^l x_k \in B(x_k, \epsilon) \text{ for all } l \in \mathbb{N}.$$

That is, $x_q \in B(x_k, \epsilon)$ for all $q \ge k$. Hence

$$A_b\big(x_l, x_l, x_l, \dots, (x_l)_{n-1}, x_q\big) < \epsilon \text{ for all } q, l > k. \tag{9}$$

Therefore $\{x_k\}$ is a Cauchy sequence and by the completeness of $X$, there exists $u \in X$ such that $x_k \to u$ as $k \to \infty$. Moreover, $u = lim_{k\to\infty} x_{k+1} = lim_{k\to\infty} x_k = F(u)$.Thus, $F$ has $u$ as a fixed point.

Now we prove the uniqueness of the fixed point for $F$. Since $\varphi(t) = \psi^m(t) < t$ for any $t > 0$, let $u$ and $v$ be two fixed points of $F$.

$$\begin{aligned} A_b(u, u, u, u, \dots, (u)_{n-1}, v) &\le A_b(Fu, Fu, Fu, Fu, \dots, (Fu)_{n-1}, Fv) \\ &\le \psi^m[A_b(u, u, u, u, \dots, (u)_{n-1}, v)] \\ &= \varphi[A_b(u, u, u, u, \dots, (u)_{n-1}, v)] \\ &\le A_b(u, u, u, u, \dots, (u)_{n-1}, v) \end{aligned}$$

This implies that $A_b(u, u, u, u, \dots, (u)_{n-1}, v) = 0 \Rightarrow u = v$ and hence, $F$ has unique fixed point in $X$.

**Theorem 4.3** Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a $A_b$-metric $A_b$on $X$ such that $(X, A_b, s)$ is a complete $A_b$-metric space. Let $f: X \to X$ be an continuous non-decreasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose also that for all $x^1, x^2, x^3, \dots, x^n \in X$ with $x^1 \preceq x^2 \preceq x^3 \preceq \cdots \preceq x^n$;

$$A_b(fx^1, fx^2, fx^3, \dots, fx^n) \le \alpha \prod_{i=1}^{n} A_b(x^i, x^i, x^i, \dots, fx^i)\,([A_b(x^1, x^2, x^3, \dots, x^n)]^{n-1})^{-1} + \beta A_b(x^1, x^2, x^3, \dots, x^n) \tag{10}$$

where $\alpha + s^2\beta < 1$. Then $f$ has a fixed point. Moreover, the set of fixed points of $f$ is well ordered if and only if $f$ has one and only one fixed point.

**Proof** Starting with the given $x_0 \in X$, put $x_k = f^k x_0 = f x_{k-1}$. Since $x_0 \preceq f x_0$ and $f$ is an increasing function, we obtain by induction that

$$x_0 \preceq f x_0 \preceq f^2 x_0 \preceq \cdots \dots \dots . . \preceq f^k x_0 \preceq f^{k+1} x_0 \preceq \cdots \dots \dots .... \tag{11}$$

Considering the sequence and using (10), we have

$$\begin{aligned} A_b(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1}) &= A_b(fx_{k-1}, fx_{k-1}, fx_{k-1}, \dots., (fx_{k-1})_{n-1}, fx_k) \\ &\le \alpha\{A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, fx_{k-1}) \\ &\times A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, fx_{k-1}) \times \dots.. \\ &\times \left(A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, fx_{k-1})\right)_{n-1} \\ &\times A_b(x_k, x_k, x_k, \dots., (x_k)_{n-1}, fx_k)\} \\ &\times ([A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k)]^{n-1})^{-1} \\ &+\beta A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k) \\ &= \alpha\left(A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k)\right)^{n-1} \\ &\times A_b(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1})\} \\ &\times ([A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k)]^{n-1})^{-1} \\ &+\beta A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k) \end{aligned}$$

The last inequality gives

$$A_b(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_{k+1}) \le \lambda A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., (x_{k-1})_{n-1}, x_k) \tag{12}$$

for all $k \in \mathbb{N} \cup \{0\}$ where $\lambda = \frac{\beta}{1-\alpha} < \frac{1}{s^2}$. By Lemma 4.1, $\{x_k\}$ is a Cauchy sequence in $X$. Since X is a complete $A_b$-metric space, there exists $x^\star \in X$ such that $\lim_{k\to+\infty} x_k \to x^\star$. If $f$ is $A_b$-continuous, then

$$fx^\star = f(\lim_{k\to+\infty} x_k) = \lim_{k\to+\infty} fx_k = \lim_{k\to+\infty} x_{k+1} = x^\star \tag{13}$$

This implies that $x^\star$ is a fixed point of $f$.

Finally, suppose that the set of fixed points of $f$ is well ordered. Assume, to the contrary, that $u$ and $v$ are two distinct fixed points of $f$.

$$\begin{aligned} A_b(fu, fu, fu, fu, , \dots, (fu)_{n-1}, fv) &\le \alpha\{A_b(u, u, u, u, , \dots, (u)_{n-1}, fu) \\ &\times A_b(u, u, u, u, , \dots, (u)_{n-1}, fu) \times \dots .... \\ &\times \left(A_b(u, u, u, u, , \dots, (u)_{n-1}, fu)\right)_{n-1} \\ &\times A_b(v, v, v, v, \dots, (v)_{n-1}, fv)\} \\ &\times ([u, u, u, u, , \dots, (u)_{n-1}, v]^{n-1})^{-1} \\ &+\beta A_b(u, u, u, u, , \dots, (u)_{n-1}, v) \end{aligned}$$

Thus, we get

$$A_b(u, u, u, u, , \dots, (u)_{n-1}, v) \le \beta A_b(u, u, u, u, , \dots, (u)_{n-1}, v)$$

a contradiction. Hence, $f$ has a unique fixed point. The converse is trivial.

## 5 Results under Geraghty-type Conditions

In 1973, Geraghty [25] proved a fixed point result, generalizing the Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. Fixed point results of this kind in $b$-metric spaces were obtained by Dukić et al. in [26].

Following [26-28], for a real number $s \geq 1$ let $\mathcal{F}_s$ denote the class of all functions $\beta : [0, \infty) \rightarrow [0, \frac{1}{s})$ satisfying the following condition:

$$\beta(t_k) \rightarrow \frac{1}{s} \text{ as } k \rightarrow \infty \text{ implies } t_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Now, we have the following fixed point theorem in $A_b$-metric space.

**Theorem 5.1** Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a $A_b$-metric $A_b$ on $X$ such that $(X, A_b, s)$ is a $A_b$-complete $A_b$-metric space. Let $f: X \rightarrow X$ be a non- decreasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose also that for all $x^1, x^2, x^3, \dots, x^n \in X$ with $x^1 \preceq x^2 \preceq x^3 \preceq \cdots \preceq x^n$;

$$sA_b(fx^1, fx^2, fx^3, \dots, fx^n) \leq \beta\big(A_b(x^1, x^2, x^3, \dots, x^n)\big)M(x^1, x^2, x^3, \dots, x^n) \tag{14}$$

where

$$M(x^1, x^2, x^3, \dots, x^n) = \max\left\{A_b(x^1, x^2, x^3, \dots, x^n), \frac{\prod_{i=1}^{n} A_b(x^i, x^i, x^i, \dots, fx^i)}{1 + \left(A_b(x^1, x^2, x^3, \dots, x^n)\right)^{n-1}}\right\}$$

and $\beta \in \mathcal{F}_s$. Then $f$ has a fixed point. Moreover, the set of fixed points of $f$ is well ordered if and only if $f$ has one and only one fixed point.

**Proof** Starting with the given $x_0 \in X$, put $x_k = f^k x_0 = fx_{k-1}$. Since $x_0 \preceq fx_0$ and $f$ is an increasing function, we obtain by induction that

$$x_0 \preceq fx_0 \preceq f^2 x_0 \preceq \cdots \dots \dots . . \preceq f^k x_0 \preceq f^{k+1} x_0 \preceq \cdots \dots \dots \dots$$

Considering the sequence and using (14), we have

$$\begin{aligned} sA_b(x_k, x_k, x_k, \dots ., (x_k)_{n-1}, x_{k+1}) &= sA_b(fx_{k-1}, fx_{k-1}, fx_{k-1}, \dots ., (fx_{k-1})_{n-1}, fx_k) \\ &\leq \beta\big(A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k)\big) \\ &\times M(x_{k-1}, x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k) \\ &\leq \frac{1}{s} A_b(x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k) \\ &\leq A_b(x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k) \end{aligned} \tag{15}$$

because

$$\begin{aligned} &M(x_{k-1}, x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k) \\ &\quad = \max\Big\{A_b(x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k), \\ &\quad \frac{\left(A_b(x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k)\right)^{n-1} A_b(x_k, x_k, \dots ., (x_k)_{n-1}, x_{k+1})}{1 + \left(A_b(x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k)\right)^{n-1}}\Big\} \\ &\quad = A_b(x_{k-1}, x_{k-1}, \dots ., (x_{k-1})_{n-1}, x_k) \end{aligned}$$

Therefore, the sequence $\{A_b(x_k, x_k, x_k, \dots ., (x_k)_{n-1}, x_{k+1})\}$ is decreasing. Then there exists $r \geq 0$ such that

$$\lim_{k\to+\infty} A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1}) = r. \tag{16}$$

We claim $r = 0$.

Suppose that $r > 0$. Then letting $k \to +\infty$, from (15) we have

$$\frac{1}{s} r \le sr \le \lim_{k\to+\infty} \beta\big(A_b(x_{k-1},x_{k-1},x_{k-1},\dots,(x_{k-1})_{n-1},x_k)\big)\, r \le r.$$

So we have $\lim_{k\to+\infty} \beta\big(A_b(x_{k-1},x_{k-1},x_{k-1},\dots,(x_{k-1})_{n-1},x_k)\big) \ge \frac{1}{s}$ and since $\beta \in \mathcal{F}_s$, we deduce that $\lim_{k\to+\infty} A_b(x_{k-1},x_{k-1},x_{k-1},\dots,(x_{k-1})_{n-1},x_k) = 0$, which is contradiction. Hence,

$$\lim_{k\to+\infty} A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1}) = r = 0. \tag{17}$$

Now

$$\begin{aligned}
A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m) &\le s(n-1)A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1}) \\
&\quad + sA_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{k+1}) \\
&\le s(n-1)A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1}) \\
&\quad + s^2A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_m) \\
&\le s(n-1)A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1}) \\
&\quad + s^3(n-1)A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1}) \\
&\quad + s^3A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1}) \\
&\le s(n-1)A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1}) \\
&\quad + s^2(n-1)\beta\big(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)\big) \\
&\quad \times M(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m) \\
&\quad + s^3A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1})
\end{aligned} \tag{18}$$

Letting $k, m \to +\infty$ in the above inequality and applying (17), we have

$$\begin{aligned}
&\lim_{k,m\to+\infty} A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m) \\
&\qquad \le s^2(n-1)\lim_{k,m\to+\infty}\big(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)\big) \\
&\qquad \times \lim_{k,m\to+\infty} M(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)
\end{aligned} \tag{19}$$

Here

$$\begin{aligned}
A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m) &\le M(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m) \\
&= \max\Big\{A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m), \\
&\qquad \frac{\big(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},fx_k)\big)^{n-1} \times A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},fx_m)}{1+\big(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)\big)^{n-1}}\Big\}
\end{aligned}$$

Letting $k, m \to +\infty$ in the above inequality, we get

$$\begin{aligned}
&\lim_{k,m\to+\infty} M(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m) \\
&\qquad = \lim_{k,m\to+\infty} A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)
\end{aligned} \tag{20}$$

Hence from (19), we have

$$\begin{aligned}
&\lim_{k,m\to+\infty} A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m) \\
&\qquad \le s^2(n-1)\lim_{k,m\to+\infty}\beta\big(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)\big) \\
&\qquad \times \lim_{k,m\to+\infty} A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)
\end{aligned} \tag{21}$$

Now we claim that $\lim_{k,m\to+\infty} A_b(x_k, x_k, x_k, \dots\dots\dots, (x_k)_{n-1}, x_m) = 0$ . If, to the contrary, $\lim_{k,m\to+\infty} A_b(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_m) \neq 0$, then we get

$$\lim_{k,m\to+\infty}\beta\big(A_b(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x_m)\big) \geq \frac{1}{s^2(n-1)} \geq \frac{1}{s^2}$$

which is contradiction. Consequently, $\{x_k\}$ is a $A_b$-Cauchy sequence in $X$. Since $(X, A_b, s)$ is $A_b$-complete, the sequence $\{x_k\}$ $A_b$-converges to some $x^\star \in X$, that is, $\lim_{k\to+\infty} A_b(x_k, x_k, x_k, \dots., (x_k)_{n-1}, x^\star) = 0$. Now we show that $x^\star$ is a fixed point of $f$. If $f$ is $A_b$-continuous, then

$$fx^\star = f\left(\lim_{k\to+\infty} x_k\right) = \lim_{k\to+\infty} fx_k = \lim_{k\to+\infty} x_{k+1} = x^\star$$

This implies that $x^\star$ is a fixed point of $f$.

Finally, suppose that the set of fixed points of $f$ is well ordered. Assume, to the contrary, that $u$ and $v$ are two distinct fixed points of $f$.

$$\begin{aligned} sA_b(u, u, u, \dots., (u)_{n-1}, v) &= sA_b(fu, fu, fu, fu, , \dots, (fu)_{n-1}, fv) \\ &\leq \beta\big(A_b(u, u, u, \dots., (u)_{n-1}, v)\big)M(u, u, u, \dots., (u)_{n-1}, v) \\ &= \beta\big(A_b(u, u, u, \dots., (u)_{n-1}, v)\big)A_b(u, u, u, \dots., (u)_{n-1}, v) \end{aligned} \tag{22}$$

Because

$$\begin{aligned} M(u, u, u, \dots., (u)_{n-1}, v) &= \max\Bigg\{A_b(u, u, u, \dots., (u)_{n-1}, v), \\ &\qquad \frac{\big(A_b(u,u,u,\dots.,(u)_{n-1},fu)\big)^{n-1} \times A_b(v,v,v,\dots.,(v)_{n-1},fv)}{1+\big(A_b(u,u,u,\dots.,(u)_{n-1},v)\big)^{n-1}}\Bigg\} \\ &= \max\{A_b(u, u, u, \dots., (u)_{n-1}, v), 0\} \\ &= A_b(u, u, u, \dots., (u)_{n-1}, v) \end{aligned}$$

Then we get

$$\beta\big(A_b(u, u, u, \dots., (u)_{n-1}, v)\big) \geq \frac{1}{s}$$

a contradiction. Hence, $f$ has a unique fixed point. The converse is trivial.

Note that the continuity of $f$ in Theorem 5.1 can be replaced by certain property of the space itself.

**Theorem 5.2** Under the hypotheses of Theorem 5.1, without the $A_b$-continuity assumption, assume that whenever $\{x_k\}$ is a non-decreasing sequence in $X$ such that $x_k \to x^\star$, one has $x_k \preceq x^\star$ for all $k \in \mathbb{N}$. Then $f$ has a fixed point. Moreover, the set of fixed points of $f$ is well ordered if and only if $f$ has one and only one fixed point.

**Proof** Repeating the proof of Theorem 5.1, we construct an increasing sequence $\{x_k\}$ in $X$such that $x_k \to x^\star \in X$. Using the assumption on $X$ we have $x_k \preceq x^\star$. Now, we show that $x^\star = fx^\star$. By (14) and Lemma 3.11, we have

$$\begin{aligned} s^2\left[\frac{1}{s^2}A_b(x^\star, x^\star, x^\star, x^\star, \dots, (x^\star)_{n-1}, fx^\star)\right] &\leq s^2 \lim_{k\to+\infty}\sup A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, fx^\star) \\ &= s^2 \lim_{k\to+\infty}\sup A_b(fx_k, fx_k, fx_k, \dots, (fx_k)_{n-1}, fx^\star) \\ &\leq s \lim_{k\to+\infty}\sup \beta\big(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^\star)\big) \\ &\quad \lim_{k\to+\infty}\sup M(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^\star) \end{aligned} \tag{23}$$

Where

$$\lim_{k\to+\infty} M(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^\star) = \lim_{k\to+\infty} \max\{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^\star), \frac{(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},fx_k))^{n-1} \times A_b(x^\star,x^\star,x^\star,\dots,(x^\star)_{n-1},fx^\star)}{1+(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x^\star))^{n-1}}\}$$
$$= \lim_{k\to+\infty} \max\{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x^\star), \frac{(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1}))^{n-1} \times A_b(x^\star,x^\star,x^\star,\dots,(x^\star)_{n-1},fx^\star)}{1+(A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x^\star))^{n-1}}\}$$
$$= 0$$

Therefore we deduce that $A_b(x^\star, x^\star, x^\star, x^\star, \dots, (x^\star)_{n-1}, fx^\star) = 0$, hence $fx^\star = x^\star$. The proof of uniqueness is the same as in Theorem 5.1.

# 6 Fixed Point Theorems for Expansion Mapping

In this section, first we prove some fixed point theorem satisfying expansive condition by considering surjective self-mapping in the context of $A_b$- metric space

**Theorem 6.1** Let $(X, A_b, s)$ be a complete $A_b$-metric space with the coefficient $s \geq 1$. Assume that the mapping $T: X \to X$ is surjection and satisfies

$$A_b(Tx^1, Tx^2, Tx^3, \dots . Tx^{n-1}, Tx^n) \geq \lambda A_b(x^1, x^2, x^3, \dots . x^{n-1}, x^n) \tag{24}$$

$\forall x^1, x^2, x^3, \dots . x^{n-1}, x^n \in X$, where $\lambda > s^2$. Then $T$ has a fixed point.

**Proof** Let $x_0 \in X$, since T is surjection on $X$, then there exists $x_1 \in X$ such that $x_0 = Tx_1$. By continuing this process, we get

$$x_k = Tx_{k+1}, \ \forall\, k \in \mathbb{N} \cup \{0\}. \tag{25}$$

If $A_b(x_{m-1}, x_{m-1}, x_{m-1}, \dots, x_{m-1}, x_m) = 0$ for some $m$, then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so $x_m$ is a fixed point of $T$. Without loss of generality, we can suppose that $A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_{k-1} x_n) > 0$, that is, $x_k \neq x_{k-1}$ for every $k$. Consider from (24), we have

$$A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_k) = A_b(Tx_k, Tx_k, Tx_k, \dots, Tx_{k+1})$$
$$\geq \lambda A_b(x_k, x_k, x_k, \dots, x_{k+1})$$

and so

$$A_b(x_k, x_k, x_k, \dots, x_{k+1}) \leq \frac{1}{\lambda} A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_k)$$
$$= hA_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, x_k) \tag{26}$$

for all $k \in \mathbb{N}$, where $h = \frac{1}{\lambda} < \frac{1}{s^2}$. By Lemma 4.1,$\{x_k\}$ is a Cauchy sequence in $X$. Since X is a complete $A_b$-metric space, there exists $x^\star \in X$ such that $x_n \to x^\star$ as $k \to +\infty$. Now since $T$ is surjective map. So there exists a point $p$ in $X$ such that $x^\star = Tp$. Consider from (24), we have

$$A_b(x_k, x_k, x_k, \dots, x_k, x^\star) = A_b(Tx_{k+1}, Tx_{k+1}, Tx_{k+1}, \dots, Tx_{k+1}, Tp)$$
$$\geq \lambda A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, p)$$

Taking limit as $k \to +\infty$ in the above inequality, we get

$$0 = \lim_{k\to+\infty} A_b(x_k, x_k, x_k, \dots, x_k, x^\star) \geq \lambda \lim_{n\to\infty} A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, x_{k+1}, p)$$

which implies that

$$\lim_{n\to+\infty} A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots., x_{k+1}, p) = 0. \tag{27}$$

Thus $x_{n+1} \to p$ as $k \to +\infty$. By Lemma 3.9, we get $x^\star = p$. Hence $x^\star$ is a fixed point of $T$.

Finally, assume $x^\star = y^\star$ is also another fixed point of $T$. From (24), we get

$$\begin{aligned} A_b(x^\star, x^\star, x^\star, x^\star, \dots., (x^\star)_{n-1}, y^\star) &= A_b(Tx^\star, Tx^\star, Tx^\star, Tx^\star, \dots., (Tx^\star)_{n-1}, y^\star) \\ &\geq \lambda A_b(x^\star, x^\star, x^\star, x^\star, \dots., (x^\star)_{n-1}, y^\star) \end{aligned}$$

This is true only when $A_b(x^\star, x^\star, x^\star, x^\star, \dots., (x^\star)_{n-1}, y^\star) = 0$. So $x^\star = y^\star$. Hence $T$ has a unique fixed point in $X$.

**Corollary 6.2** Let $(X, A_b)$ be a complete $A_b$-metric space with the coefficient $s \geq 1$ and $T: X \to X$ be a surjection. Suppose that there exist a positive integer $k$ and a real number $\lambda > s^2$ such that

$$A_b\left(T^k(x^1), T^k(x^2), \dots, T^k(x^{n-1}), T^k(x^n)\right) \geq \lambda A_b(x^1, x^2, \dots. x^{n-1}, x^n) \tag{28}$$

$\forall x^1, x^2, x^3, \dots. x^{n-1}, x^n \in X$. Then $T$ has a fixed point.

**Proof** From Theorem 6.1, $T^k$ has a fixed point $x^\star$. But $T^k(Tx^\star) = T(T^k x^\star) = Tx^\star$, So $Tx^\star$ is also a fixed point of $T^k$. Hence $Tx^\star = x^\star, x^\star$ is a fixed point of $T$. Since the fixed point of $T$ is also fixed point of $T^k$, the fixed point of $T$ is unique.

Now, motivated by the work in Jain et al. [27-28], we give the following.

Let $\Psi_{\mathcal{B}}^L$ denote the class of those function $\mathcal{B}: (0, \infty) \to (L^2, \infty)$ which satisfy the condition $\mathcal{B}(t_k) \to (L^2)^+ \Rightarrow k \to 0$, as $k \to \infty$, where $L > 0$.

**Theorem 6.3** Let $(X, A_b)$ be a complete $A_b$-metric space with $s \geq 1$. Assume that the mapping $T: X \to X$ is surjection and satisfies

$$A_b(Tx^1, Tx^2, Tx^3, \dots, Tx^n) \geq \mathcal{B}\left(A_b(x^1, x^2, x^3, \dots, x^n)\right) A_b(x^1, x^2, x^3, \dots, x^n) \tag{29}$$

$\forall\, x^1, x^2, x^3, \dots, x^n \in X$, where $\mathcal{B} \in \Psi_{\mathcal{B}}^s$. Then $T$ has a fixed point.

**Proof** Pick$x_0 \in X$. Since T is surjective, choose $x_1 \in X$ such that$Tx_1 = x_0$. Inductively, we can define a sequence $\{x_k\} \subset X$ such that $x_k = Tx_{k+1}$, $\forall\, n \in \mathbb{N} \cup \{0\}$. If $A_b(x_{m-1}, x_{m-1}, x_{m-1}, \dots, x_{m-1}, x_m) = 0$ for some $m$, then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so $x_m$ is a fixed point of $T$. Without loss of generality, we can suppose that $A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots., x_{k-1} x_n) > 0$, that is, $x_k \neq x_{k-1}$ for every $k$. Consider

$$\begin{aligned} &A_b(x_{k-1}, x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\ &\quad = A_b(Tx_k, Tx_k, Tx_k, \dots, (Tx_k)_{n-1}, Tx_{k+1}) \end{aligned}$$

Now by (29) and definition of the sequence

$$\begin{aligned} &A_b(x_{k-1}, x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k) \\ &\quad = A_b(Tx_k, Tx_k, Tx_k, Tx_k, \dots, (Tx_k)_{n-1}, Tx_{k+1}) \\ &\quad \geq \mathcal{B}\left(A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})\right) \\ &\quad \times A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad \geq s^2 A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \end{aligned}$$

$$\geq A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \tag{30}$$

Thus the sequence $\{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})\}$ is a decreasing sequence in $\mathbb{R}^+$ and so there exists $\epsilon \geq 0$ such that

$$\lim_{n\to\infty} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) = \delta \tag{31}$$

Let us prove that $\delta = 0$. Suppose to the contrary that $\delta > 0$. By (30) we can deduce that

$$s^2 \frac{A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)}{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})} \geq \frac{A_b(x_{k-1}, x_{k-1}, x_{k-1}, \dots, (x_{k-1})_{n-1}, x_k)}{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})} \geq \mathcal{B}\big(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})\big) \geq s^2 \tag{32}$$

By taking limit as $k \to +\infty$ in the above inequality, we have

$$\lim_{k\to+\infty} \mathcal{B}\big(A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})\big) = s^2 \tag{33}$$

Hence by definition of $\mathcal{B}$, we have

$$\delta = \lim_{k\to+\infty} A_b(x_k, x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) = 0 \tag{34}$$

which is a contradiction. Hence $\delta = 0$. Now, we shall show that for $k, m \in \mathbb{N}$,

$$\lim_{k,m\to+\infty} \sup A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) = 0 \tag{35}$$

Suppose to the contrary that $\lim_{k,m\to+\infty} \sup A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) > 0$.

By (29), we have

$$\begin{aligned} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) &= A_b(Tx_{k+1}, Tx_{k+1}, Tx_{k+1}, \dots, (Tx_{k+1})_{n-1}, Tx_{m+1}) \\ &\geq \mathcal{B}\big(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1})\big) \\ &\times A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}) \end{aligned}$$

That is,

$$\frac{A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m)}{\mathcal{B}\big(A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1})\big)} \geq A_b(x_{k+1}, x_{k+1}, x_{k+1}, \dots, (x_{k+1})_{n-1}, x_{m+1}) \tag{36}$$

From (Ab3), we have

$$\begin{aligned} A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_m) &\leq s[A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad +A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad +A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) + \cdots \\ &\quad +\big(A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1})\big)_{n-1} \\ &\quad +A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+1})] \\ &= s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad +sA_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{k+1}) \\ &\leq s(n-1)A_b(x_k, x_k, x_k, \dots, (x_k)_{n-1}, x_{k+1}) \\ &\quad +s^2[A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) \\ &\quad +A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) \\ &\quad +A_b(x_m, x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}) + \cdots \end{aligned}$$

$$\begin{aligned}&+\big(A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1})\big)_{n-1}\\&+A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})]\\&=s(n-1)A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1})\\&+s^2(n-1)A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1})\\&+s^2A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})\\&\le s(n-1)A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1})\\&+s^2(n-1)A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1})\\&+s^2\frac{A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)}{\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})\big)}\end{aligned} \tag{37}$$

The last inequality gives,

$$\begin{aligned}\left(1-\frac{s^2}{\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})\big)}\right)&A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)\\&\le s(n-1)A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1})\\&+s^2(n-1)A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1})\end{aligned}$$

That is,

$$\begin{aligned}\left(1-\frac{s^2}{\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})\big)}\right)&\le s(n-1)\frac{A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1})}{A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)}\\&+s^2(n-1)\frac{A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1})}{A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)}\end{aligned}$$

Thus,

$$\begin{aligned}&\left(1-\frac{s^2}{\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})\big)}\right)^{-1}\\&\qquad\le\frac{1}{(n-1)}\left(\frac{A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)}{sA_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1})+s^2A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1})}\right)\end{aligned} \tag{38}$$

By taking limit as $k,m\to+\infty$ in the above inequality, since

$$\lim_{k,m\to+\infty}\sup A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)>0$$

and

$$\begin{aligned}\delta=0&=\lim_{k\to+\infty}A_b(x_k,x_k,x_k,\dots,(x_k)_{n-1},x_{k+1})\\&=\lim_{m\to+\infty}A_b(x_m,x_m,x_m,\dots,(x_m)_{n-1},x_{m+1}),\end{aligned}$$

then we obtain

$$\lim_{k,m\to+\infty}\left(1-\frac{s^2}{\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})\big)}\right)^{-1}=+\infty \tag{39}$$

which implies that

$$\lim_{k,m\to+\infty}\sup\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})\big)=(s^2)^+ \tag{40}$$

and so by definition of $\mathcal{B}$, we have

$$\lim_{k,m\to+\infty}\sup A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},x_{m+1})=0 \tag{41}$$

which is a contradiction. Hence,

$$\lim_{k,m\to+\infty}\sup A_b(x_k,x_k,x_k,x_k,\dots,(x_k)_{n-1},x_m)=0. \tag{42}$$

and so, $\{x_k\}$ is a Cauchy sequence. Since $X$ is a complete $A_b$-metric space, the sequence $\{x_k\}$ in $X$ converges to $x^\star \in X$. so that

$$\lim_{k\to+\infty} A_b(x_k,x_k,x_k,x_k,\dots,(x_k)_{n-1},x^\star)=0. \tag{43}$$

As $T$ is surjective, so there exists $p\in X$ such that $x^\star = Tp$. Let us prove that $x^\star = p$. Suppose to the contrary that $x^\star \neq p$. Then by (29), we have

$$\begin{aligned} A_b(x_k,x_k,x_k,x_k,\dots,(x_k)_{n-1},x^\star) &= A_b(Tx_{k+1},Tx_{k+1},Tx_{k+1},\dots,(Tx_{k+1})_{n-1},Tp)\\ &\geq \mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},p)\big)\\ &\times A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},p) \end{aligned} \tag{44}$$

By Taking limit as $k\to+\infty$ in the above inequality and applying Lemma 3.11, we obtain

$$\begin{aligned} 0 &= \lim_{k\to+\infty} A_b(x_k,x_k,x_k,x_k,\dots,(x_k)_{n-1},x^\star)\\ &\geq \lim_{k\to+\infty}\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},p)\big)\\ &\times \lim_{k\to\infty} A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},p)\\ &\geq \frac{1}{s}\lim_{k\to+\infty}\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},p)\big)\\ &\times \lim_{k\to\infty} A_b(x,x,x,\dots,(x)_{n-1},p) \end{aligned} \tag{45}$$

and hence,

$$\lim_{k\to+\infty}\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},p)\big)=0$$

which is a contradiction. Indeed,

$$\lim_{n\to+\infty}\mathcal{B}\big(A_b(x_{k+1},x_{k+1},x_{k+1},\dots,(x_{k+1})_{n-1},p)\big)\geq s^2.$$

Since $\mathcal{B}(t) > s^2$ for all $t\in[0,+\infty)$, therefore$x^\star = p$. Hence $x^\star = Tp = Tx^\star$.

## 7 Conclusion

In this paper, we introduced the notion of $A_b$-metric space and to studied its basic topological properties. We established some fixed point theorems under different contraction and expansion type conditions in the setting of $A_b$-metric space and partially ordered $A_b$-metric space.Our results generalize and extend various results in [3, 24-28].

## Competing Interests

Authors have declared that no competing interests exist.

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