



A Characterization of a Special Class of Operator Matrices

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Authors' contributions

This work was carried out in collaboration among all the authors. Author OMO designed the study, performed the mathematical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors WJW and SAN managed the analyses of the study and improvement of some of the results. All authors read and approved the final manuscript.

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Abstract

The classification of operator matrices has attracted a great deal of research in the recent past. To the class of such matrices belongs; the Normal, Binormal, Hypernormal, Hamadard, Toeplitz, Pythagorean matrix operators among others. These operator matrices demonstrate many classical properties on dealing with them in connection to algebraic structural properties. In the case of pythagorean matrices in which the column entries are the entries of the triplets(right triangle) of consecutive integers, the shift operator matrix preserves the order and nature of the original matrix. These classes of operators have been studied before to a fair extent, however, from the documented literature, normal operator matrices that result from matrix products in direct sums of Hilbert spaces have not been characterized before. In particular, there is no mention in literature of a classification of normal matrices resulting from a combination of Hadamard and Khatri-Rao decompositions on Hilbert spaces. On the other hand, the matrix products have

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found applications in Information Science(signal sensing), Coding Theory(quantum error computation) among other areas. In this paper, we characterize a special class of normal operator matrices of pythagorean type, which are newly constructed as the Khatri-Rao(which generalizes Hadamard products) products whose entries are the block matrices of pythagorean triplets of class C^1 and extend the findings to an arbitrary C^m completed normal matrix of the same category. We provide detailed survey on the normality and subnormality conditions, positivity and boundedness, and prove new forms of numerical and spectral radii properties as well as the inherent structural relationships of the constructed matrix operator.

Keywords: Normal matrices; subnormality/hyponormality; numerical (resp. spectral) radii; positivity.

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1 Introduction

We denote by H – the complex separable Hilbert space, $B(H)$ – the Banach space of bounded linear operators, M_n – the set of special classes of normal matrices constructed and embedded in the class C^1 , ω – the numerical radius of the operator defined by either T or $A \in M_n \in C^1$, γ – the spectral radius of an operator. Other notations are standard and can be obtained from the references or will be defined from time to time.

A lot of classes of Hilbert spaces operators are defined around the notion of Normal operators. For example, in [1], Paul Halmos, motivated by the successful development of Normal operators, extended the notion of subnormality and hyponormality for the bounded Hilbert space operators in an attempt to enrich the basic facts of the spectral theory of the Normal operators. Those classes of the operators have been the subject of much investigations for quite some time and many important developments in the operator theory have dealt with them (see for instance [2, 3, 4, 5]). As a result, certain successes have been felt, for instance the proof of the non-existence of non-trivial subspaces, construction of the functional calculus and description of the spectral picture in the cyclic case for subnormal operators among others.

A complex square matrix A is Normal if $A^*A = AA^*$ where A^* is the conjugate transpose of A . A real square matrix A is thus normal iff $A^* = A^T$ so that $A^T A = AA^T$. Such matrices are unitary equivalent to diagonal matrices and therefore, every normal matrix is diagonalizable. Unitary, Hermitian, Skew-Hermitian matrices are some of the examples of complex normal matrices. Some classes of such matrices have been studied independently and the results around them are standard (cf.[6]). Other approaches can also be used to construct new forms of normal matrix operators. In view of this, there are a number of matrix products that have proved fundamentally important in many areas such as engineering, natural and social sciences, matrix theory, statistics, system theory and other areas. The Hadamard, Khatri-Rao, Kronecker and several related non-simple matrix products and their properties have been studied by Liu and Trenkler [7]. They gave an equality connection between the Hadamard and Kronecker products and observed that the Khatri-Rao product generalizes the notion of Hadamard product for partitioned matrices hence exhibiting wider applications. In signal processing, the space-time coding techniques exploit the spatial diversity afforded by multiple transmitting and receiving antennas to achieve reliable transmission in scattering-rich environments. Sidiropoulos and Budampati [8] proposed a broad new class of space-time codes based on the Khatri-Rao product, KRST codes, for short. They reported that KRST codes are linear block codes designed to provide several benefits, which yield better performance than linear dispersion codes at high signal-to-noise ratio and than linear constellation precoding codes using a lower order constellation. Consider the multi-antenna system with M transmitting antennas and N receiving antennas. The wireless channel is assumed to be quasi-static and

fading. The discrete-time baseband-equivalent model for the received data is given (when the channel is constant for at least K channel uses) by

$$X = \sqrt{\frac{\rho}{M}}HC + W,$$

where X is the $N \times K$ received signal matrix, C is the $M \times K$ transmitted code matrix, W is the $N \times K$ additive noise matrix, H is the $N \times M$ channel matrix which has i.i.d. $N(0,1)$ entries being mutually independent from X and W , and ρ is the signal-to-noise ratio. The model can be improved further and applied the channel assumed to be constant for block time T to obtain the following (noiseless) vectorized model.

Based on the Khatri-Rao product, Wang et' al [9] considered a similar data model and proposed a novel Khatri-Rao unitary space-time modulation design. Their idea was to use the Khatri-Rao product to obtain a decomposition result to find a simplified maximum likelihood detection algorithm for their design. Upon the decomposition, the new detector needs to perform only a vector multiplication, instead of a matrix multiplication which the original detector needs to perform. As reported, the new design does not require any computer search and can be applied to any number of transmitting antennas, among other improvements. Despite these marvelous attempts to explore the applications of matrix products, the majority of applications of the Khatri-Rao product is still based on only the column-wise partitioned situation.

Arnon and Patrawutt [10] have provided a beautiful extension of the works in [7, 8, 9] generalized the tensor product of operators to the Khatri-Rao product of operator matrices acting on a direct sum of Hilbert spaces. They investigated fundamental properties of this operator product. Algebraically, this product is compatible with the addition, the scalar multiplication, the adjoint operation, and the direct sum of operators. By introducing suitable operator matrices, they proved that there is a unital positive linear map taking the Tracy-Singh product $A \boxtimes B$ to the Khatri-Rao product $A \boxdot B$. Hence, the Khatri-Rao product can be viewed as a generalization of the Hadamard product of operators. It can be noted that this product is closed and the resultant matrix can be associated to a normal matrix, however, the results of [10] do not present complete classification. Motivated by these findings, this paper addresses a characterization of certain spectral properties of a class of normal operator matrix related to the Khatri-Rao product matrix but uniquely constructed such that the blocks are made up of entries which are pythagorean triplets.

Further, Hirzallah et' al [11, 12], Kittaneh [13] and Hou [14] have studied the numerical radii properties of certain unclassified operator matrices while Yamazaki [15] has set the upper and lower bounds of the numerical radii inequalities conditions while Aupetit [2] has characterized the normal matrix operators by their exponents. Motivated by these developments, based on the constructions in [10], we formulate and classify a class of Normal matrices called a special class of O.M.O Normal matrices which are constructed as the Khatri-Rao products whose entries are the block matrices of pythagorean triplets of class C^1 , characterize them by the numerical radii properties, the relationship between the numerical and spectral radii as well as the necessities for their positivity and boundedness.

2 Fundamental Principles

Proposition 2.1. *An operator $T \in B(H)$ is normal if and only if $\|Tx\| = \|T^*x\|$ for every $x \in H$*

Proof. Since the zero operator $s \in B(H)$ is the unique operator with the property $\langle sx, x \rangle = 0$ for every $x \in (H)$, then the proof follows immediately from the identity. \square

Using Liouville's Theorem, Fuglede Putman in 1950 gave a nice characterization of the general normal operators. The proof to his result was however not given.

Theorem 2.1 (Fuglede). *An operator $T \in B(H)$ is normal if and only if $T^*s = sT^*$ for every operator $s \in B(H)$ such that $Ts = sT$.*

Proof. Let T be an operator in $B(H)$. For every positive integer n , let $T_n = T + \frac{T}{1!} + \frac{T^2}{2!} + \dots + \frac{T^n}{n!}$, clearly T_n is a bounded operator in $B(H)$ and the sequence $(T_n)_n$ converges in $B(H)$ to an invertible operator denoted by e^T and its inverse is e^{-T} i.e $(e^T)^{-1} = e^{-T}$. In addition, for an operator $s \in B(H)$ that commutes with T , we have that $e^T s = s e^T$ and consequently $e^{T+s} = e^T e^s = e^s e^T$. Let s be an operator in $L(H)$ such that $Ts = sT$. Then, it follows that $s = e^{-iZT} s e^{iZT}$ for every $Z \in \mathbb{C}$ and so, for every $Z \in \mathbb{C}$, $e^{iZT^*} s e^{iZT^*} = e^{-iZT^*} e^{-i\bar{Z}T} s e^{i\bar{Z}T} e^{iZT^*} = e^{-i(ZT^* + \bar{Z}T)} s e^{i(ZT^* + \bar{Z}T)}$. Since the operator $e^{i(ZT^* + \bar{Z}T)}$ is unitary and its adjoint is $e^{-i(ZT^* + \bar{Z}T)}$, then $e^{-iZT^*} s e^{iZT^*} = (e^{i(ZT^* + \bar{Z}T)})^* s e^{i(ZT^* + \bar{Z}T)}$; thus the following operator valued function defined on \mathbb{C} by $\phi(Z) = e^{-iZT^*} s e^{iZT^*}$ is a bounded analytic function by Liouville's Theorem, it is a constant function. In particular its derivative ϕ' is zero so

$$\begin{aligned} \phi'(Z) &= -iT^* e^{-iZT^*} s e^{iZT^*} + e^{-iZT^*} s (iT^*) e^{iZT^*} . \\ &= -iT^* \phi(Z) + e^{-iZT^*} s e^{iZT^*} (iT^*) . \\ &= -iT^* \phi(Z) + \phi(Z)(iT^*) . \\ &= 0 . \end{aligned}$$

Hence $T^* \phi(Z) = \phi(Z) T^*$ for every $Z \in \mathbb{C}$. Thus $T^*s = sT^*$ because $\phi(0) = s$ □

Definition 2.1. (cf.[6]) An operator $T \in B(H)$ is called subnormal if it has a normal extension, that is, if there exists a normal operator s on a Hilbert space K such that H is a closed invariant subspace of s and the restriction of s to H coincides with T . A normal extension s on a Hilbert space K of a subnormal operator $T \in B(H)$ is called a minimal normal extension if there is no closed invariant subspace for s on which the restriction on s is normal operator; such an extension always exists by Zorn's Lemma and it is unique up to an invertible isometry.

Lemma 2.2. *Let $T \in B(H)$ be a subnormal operator. A normal extension s of T on a Hilbert space K is a minimal normal extension if and only if K coincides with the closure of the linear subspace generated by $\{s^{*n}x : n \in \mathbb{N}\}$*

Lemma 2.3. *Let s be a normal extension on a Hilbert space K of a subnormal operator $T \in B(H)$, then for every finite family of elements $x_1, x_2, \dots, x_n \in H$, we have :*

$$\left\| \sum_{i=1}^n s^{*i} x_i \right\| = \left\| \sum_{i=1}^n T^{*i} x_i \right\|$$

Proof. We have that

$$\begin{aligned} \left\| \sum_{i=1}^n s^{*i} x_i \right\|^2 &= \left\langle \sum_{i=1}^n s^{*i} x_i, \sum_{j=1}^n s^{*j} x_j \right\rangle \\ &= \sum_{i=1, j=1}^{n, n} \langle s^{*i} x_i, s^{*j} x_j \rangle = \sum_{1 \leq i, j \leq n} \langle s^j x_i, s^i x_j \rangle = \sum_{1 \leq i, j \leq n} \langle T^j x_i, T^i x_j \rangle \text{ because } s|_H = T \\ &= \left\| \sum_{i=1}^n T^{*i} x_i \right\|^2 . \end{aligned}$$

□

Theorem 2.4. Let $T \in B(H)$ be a subnormal operator and s be its minimal normal extension on a Hilbert space H , then $\sigma(s) \subset \sigma(T)$. In particular the minimal normal extension of an invertible subnormal operators is invertible.

Proof. To show that $\sigma(s) \subset \sigma(T)$, it suffices to prove that $s - \lambda I$ is invertible for every $\lambda \in \rho(T)$. On the other hand, $s - \lambda I$ is a minimal normal extension of the operator $s - \lambda I$, so, the assertion reduces to prove that if T is invertible, then its minimal normal extension s is invertible. It is clear that the closure of the range of s is a closed invariant subspace M for s and its adjoint s^* therefore, the restriction of s on M is normal. On the other hand, $H = TH = SH$ is contained in the range of s . Hence it follows from the minimality of s that $M = K$. So s has a dense range. Thus it suffices to prove that s is bounded from below. Let L be a linear subspace generated by the set $\{s^{*i}x_i : i \in \mathbb{N}, x \in H, \}$ we already know from lemma (*) that $\bar{L} = K$, for every finite sum $\sum_i s^{*i}x_i \in L$, we have :

$$\begin{aligned} & \|s(\sum_i s^{*i}x_i)\| = \|s^*(\sum_i s^{*i}x_i)\| \\ & = \|\sum_i s^{*i+1}x_i\| = \|\sum_i T^{*i+1}\| \|T^*(\sum_i T^{*i}x_i)\| \\ & \geq \frac{1}{\|T^{*-1}\|} \|\sum_i T^{*i}x_i\| = \frac{1}{\|T^{*-1}\|} \|\sum_i s^{*i}x_i\| \end{aligned}$$

thus $\frac{1}{\|T^{*-1}\|} \|x\| \leq \|sx\|$ for every $x \in K$ and the desired result follows. □

Theorem 2.5. For every subnormal operator $T \in B(H)$, $\|T^*x\| \leq \|Tx\|$ for every $x \in H$

Proof. We first show that $T^*x = Ps^*x$ for every $x \in H$ where s is a minimal normal extension on a Hilbert space K and P is the linear projection from $K = H \oplus H^\perp$ onto H . For every $x, y \in H$, we have:

$$\begin{aligned} \langle T^*x, y \rangle &= \langle x, Ty \rangle = \langle x, sy \rangle \text{ because } s_{IH} = T \\ &= \langle s^*x, y \rangle = \langle s^*x, py \rangle : P_{IH} = I \\ &= \langle P^*s^*x, y \rangle = \langle Ps^*x, y \rangle. \end{aligned}$$

Therefore $T^*x = Ps^*x$ for every $x \in H$ and so

$$\|T^*x\| = \|Ps^*x\| \leq \|s^*x\| = \|sx\| = \|Tx\| \text{ because } \|P\| = 1$$

□

Definition 2.2. The Normal matrix operator M is positive if and only if $\langle Mx, x \rangle \geq 0$ for all $x \in H$ and write $M \geq 0$. If $Dim_{B(H)}H = n < \infty$ then $M \in B(H)$ in such that $M \geq 0$ if it is self adjoint and its eigenvalues are non-negative.

Remark 2.1. Recall that an operator T is self adjoint if $T = T^*$ where T^* is the usual involution (adjoint) on $B(H)$ defined uniquely by the equation $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$. Now for the case where H is not necessarily finite dimensional, then eigenvalues are not useful in investigating positivity of $A = T \in M_n$. In such a case one needs to look at the spectrum of $T, \sigma(T)$ the set of complex numbers λ such that $\lambda I - T$ is not invertible. Then $T \geq 0$ if and only if $T = T^*$ and $\sigma(T) \subset [0, \infty)$.

3 The Construction and Formulation of A special O.M.O Normal Matrix

Denote by $M_{m,n}(\mathbb{C})$, the set of m-by-n complex matrices and abbreviate $M_{n,n}(\mathbb{C})$ to $M_n(\mathbb{C})$.

The Kronecker product of $M = [a_{i,j}] \in M_{m,n}(\mathbb{C})$ and $N \in M_{p,q}(\mathbb{C})$ is given by $M \hat{\otimes} N = [a_{i,j}N] \in M_{mp,nq}(\mathbb{C})$.

The Hadamard product of $M, N \in M_n(\mathbb{C})$ is defined by the entry-wise product $M \hat{\odot} N = [a_{ij}b_{ij}] \in M_n(\mathbb{C})$.

Now, let M and N be arbitrary complex matrices, partitioned into blocks A_{ij} and B_{ij} for each i, j (for the sizes of A_{ij}, B_{ij} may be different). Then the **Khatri-Rao product** for M, N is defined by $M \hat{\otimes} N = [A_{ij} \hat{\otimes} B_{ij}]_{ij}$. When M and N are non-partitioned so that each has only one block, then their Khatri-Rao product is just their Kronecker product. If M and N are entry-wise partitioned so that each block is a 1 by 1 matrix, then, their Khatri-Rao product is just their Hadamard product.

Now, let $A = M \hat{\otimes} N = [A_{ij} \hat{\otimes} B_{ij}]_{ij}$ be given. Suppose, M and N are two square matrices which are entry-wise partitioned such that each block is a 1 by 1 matrix of pythagorean type, that is, each entry is a product of the pythagorean triplets, then we can associate with a matrix A , the Hamadard product of M and N given by $A = M \hat{\odot} N = [a_{ij}b_{ij}] \in M_n(\mathbb{C})$. This matrix A is a special type of normal matrix of pythagorean type belonging to a class C^n and referred to in this paper as **A special O.M.O Normal Matrix Operator**

Remark 3.1. Let H, K be a complex separable Hilbert spaces and $B(H)$ be the Banach space of bounded linear operators on H . Suppose $A = [A_{ij}]_{i,j=1}^{m,n} \in B(H)$ and $B = [B_{ij}]_{i,j=1}^{m,n} \in B(K)$ are two special O.M.O Normal Matrices of the construction, then, merging the partitions of A such that $A = [A^{kl}]_{k,l=1}^{r,s}$; $r, s \in \mathbb{N}$, $r \leq m$, $s \leq n$, each operator A^{kl} is of $m_k \times n_l$ block in which, the

$$(g, h)^{th} \text{ block of the operator } A^{kl} \text{ is the } (u, v)^{th} \text{ block of } A, \text{ where } u = \begin{cases} g, & k = 1; \\ \sum_{i=1}^{k-1} m_i + g, & k > 1. \end{cases}$$

$$m = \sum_{k=1}^r m_k, v = \begin{cases} h, & l = 1; \\ \sum_{j=1}^{l-1} n_j + h, & l > 1. \end{cases} \text{ and } n = \sum_{l=1}^s n_l. \text{ The other matrix } B \text{ can also}$$

be repartitioned in a similar manner so that, $A \hat{\otimes} B = [A^{kl} \hat{\otimes} B^{kl}] = \begin{bmatrix} A^{11} \hat{\otimes} B^{11} & \dots & A^{1s} \hat{\otimes} B^{1s} \\ \vdots & \vdots & \vdots \\ A^{r1} \hat{\otimes} B^{r1} & \dots & A^{rs} \hat{\otimes} B^{rs} \end{bmatrix}$.

Thus each $(k, l)^{th}$ block of $A \hat{\otimes} B$ is just $A^{kl} \hat{\otimes} B^{kl}$.

From the formulation above, we clearly see that the $(u, v)^{th}$ block of $A \hat{\otimes} B$ is $A_{uv} \hat{\otimes} B_{uv}$. Thus, classes

of C^1 matrix considered in the sequel are of the form $C^1 = \begin{bmatrix} A_{11} \hat{\otimes} B_{11} & \dots & A_{1n_1} \hat{\otimes} B_{1n_1} \\ \vdots & \vdots & \vdots \\ A_{m_1 1} \hat{\otimes} B_{m_1 1} & \dots & A_{m_1 n_1} \hat{\otimes} B_{m_1 n_1} \end{bmatrix}$.

Moreover, the direct sum of such matrices $A_i \in B(H)$; $i = 1, \dots, n$ is defined to be the matrix

$$A = A_1 \oplus \dots \oplus A_n = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix}.$$

Example 3.1. Consider $A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$. We can unitarily diagonalize A by finding a unitary matrix U and a diagonal matrix D such that $A = UDU^{-1}$. To do this, we want to change the basis to one composed of orthonormal eigen vectors for $T \in B(\mathbb{C}^2)$ defined by $Tv = Av$ for all $v \in \mathbb{C}^2$.

To find such an orthonormal basis, we start by finding the eigen spaces of T

Now, $Tv = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} v$ is self adjoint or Hermitian and it can be checked that the eigen values are $\lambda = 1, 4$ by determining the zeros of the polynomial

$$p(\lambda) = (2 - \lambda)(3 - \lambda) - (1 + i)(1 - i) = \lambda^2 - 5\lambda + 4$$

so that $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. Hence

$$\mathbb{C}^2 = \text{Null}(T - I) \oplus \text{Null}(T - 4I) = \text{span}((-1 - i, 1)) \oplus \text{span}(1 + i, 2))$$

Now, applying the Gram-Schmidt procedure to each eigen space to obtain the columns of U , that is

$$\begin{aligned} A = UDU^{-1} &= \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}^{-1} \end{aligned}$$

The diagonal decomposition allows us to compute the powers and exponentials of the matrix A as follows:

$$\begin{aligned} A &= UDU^{-1} \\ A^n &= (UDU^{-1})^{-1} = UD^nU^{-1} \\ \exp(A) &= \sum_{k=0}^{\infty} \frac{1}{k(k-1)\dots(k-(k-1))} A^k \\ &= U \left(\sum_{k=0}^{\infty} \frac{1}{k(k-1)\dots(k-(k-1))} D^k \right) U^{-1} = U(\exp(D))U^{-1}. \end{aligned}$$

4 Results and Discussion

In the sequel, we consider certain examples of special O.M.O normal matrices of the construction above, characterize their numerical and spectral radii . Further, we provide a survey on their positivity and boundedness.

Proposition 4.1. Let $B(H)$ be a class of bounded linear operators, suppose $M, X \in B(H)$ such that X is an arbitrary operator and $M \in C^1$ is defined as $M = \begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix}$ where A, B, C and D are block matrices of the construction above , then M is normal and

$$w(M) = \frac{1}{\sqrt{2}} \sqrt{\left\| \begin{bmatrix} B^*B + AA^* & 0 & B^*C + AD^* \\ 0 & A^*A + D^*D + BB^* + CC^* & 0 \\ C^*B + DA^* & 0 & C^*C + DD^* \end{bmatrix} \right\|}$$

Proof. Let $M, X \in B(H)$, then

$$\begin{aligned}
 & \left| \left\langle \begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix} X, X \right\rangle \right| \leq \left| \left\langle \begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix} X, X \right\rangle \right| \cdot \left| \left\langle \begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix}^* X, X \right\rangle \right| \\
 & = \left\langle \begin{bmatrix} B^*B & 0 & B^*C \\ 0 & A^*A + D^*D & 0 \\ C^*B & 0 & C^*C \end{bmatrix}^{\frac{1}{2}} X, X \right\rangle \left\langle \begin{bmatrix} AA^* & 0 & AD^* \\ 0 & BB^* + CC^* & 0 \\ DA^* & 0 & DD^* \end{bmatrix}^{\frac{1}{2}} X, X \right\rangle \\
 & \leq \left\langle \begin{bmatrix} B^*B & 0 & B^*C \\ 0 & A^*A + D^*D & 0 \\ C^*B & 0 & C^*C \end{bmatrix} X, X \right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} AA^* & 0 & AD^* \\ 0 & BB^* & 0 \\ DA^* & 0 & DD^* \end{bmatrix} X, X \right\rangle^{\frac{1}{2}} \\
 & = \left\langle \left(\begin{bmatrix} B^*B & 0 & B^*C \\ 0 & 0 & 0 \\ C^*B & 0 & C^*C \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A^*A + D^*D & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) X, X \right\rangle^{\frac{1}{2}} \\
 & \quad \left\langle \left(\begin{bmatrix} AA^* & 0 & AD^* \\ 0 & 0 & 0 \\ DA^* & 0 & DD^* \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & BB^* + CC^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) X, X \right\rangle^{\frac{1}{2}} \\
 & \leq \frac{1}{2} \left(\left\langle \left(\begin{bmatrix} B^*B & 0 & B^*C \\ 0 & 0 & 0 \\ C^*B & 0 & C^*C \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A^*A + D^*D & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) X, X \right\rangle + \left\langle \left(\begin{bmatrix} AA^* & 0 & AD^* \\ 0 & 0 & 0 \\ DA^* & 0 & DD^* \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & BB^* + CC^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) X, X \right\rangle \right) \\
 & = \frac{1}{2} \left\langle \left(\begin{bmatrix} BB^* & 0 & B^*C \\ 0 & 0 & 0 \\ C^*B & 0 & C^*C \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A^*A + D^*D & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \right. \\
 & \quad \left. \begin{bmatrix} AA^* & 0 & AD^* \\ 0 & 0 & 0 \\ DA^* & 0 & DD^* \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & BB^* + CC^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) X, X \right\rangle \\
 & = \frac{1}{2} \left\langle \begin{bmatrix} B^*B + AA^* & 0 & B^*C + AD^* \\ 0 & A^*A + D^*D + BB^* + CC^* & 0 \\ C^*B + DA^* & 0 & C^*C + DD^* \end{bmatrix} X, X \right\rangle
 \end{aligned}$$

$= \frac{1}{2} \langle NX, X \rangle$ where N – is normal operator. Now taking the supremum for all unit vectors and considering that N is normal operator we have by $\langle X, X \rangle = \|X\|^2$ that

$$w(M) = \frac{1}{\sqrt{2}} \sqrt{\left\| \begin{bmatrix} B^*B + AA^* & 0 & B^*C + AD^* \\ 0 & A^*A + D^*D + BB^* + CC^* & 0 \\ C^*B + DA^* & 0 & C^*C + DD^* \end{bmatrix} \right\|}$$

□

Theorem 4.1. Let $M \in B(H)$ and define $M = \begin{bmatrix} X & 0 & Y \\ 0 & Y & 0 \\ Y & 0 & X \end{bmatrix} \in C^1$, then $w(M) = \max(w(X + Y), w(X - Y), w(Y))$.

Proof. Let

$$U = \begin{bmatrix} I & 0 & I \\ 0 & \sqrt{2}I & 0 \\ -I & 0 & I \end{bmatrix}.$$

Then U is unitary and

$$U M U^* = \begin{bmatrix} X + Y & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & X - Y \end{bmatrix}.$$

Since numerical radius is a weakly unitary invariant, it means that $w(UMU^*) = w(M)$. But

$$w(UMU^*) = \text{Max}(w(X + Y), w(Y), w(X - Y))$$

,that means that

$$w(M) = \text{Max}(w(X + Y), w(X - Y), w(Y))$$

□

Theorem 4.2. Let $T, M \in B(H)$ be normal matrix operators of the construction considered above and defined by $T = \begin{bmatrix} E & A & F \\ B & G & C \\ H & D & I \end{bmatrix}$ and $M = \begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix}$ where A, B, C, D, E, F, G, H and I are block matrices of pythagorean type, then

$$w(T) \leq \text{Max}(w(E), w(G), w(I)) + w(M) + \frac{\|F\| + \|H\|}{2}$$

Proof.

$$T = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ H & 0 & 0 \end{bmatrix}$$

This implies that

$$w(T) \leq w \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & I \end{bmatrix} + w \begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix} + w \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ H & 0 & 0 \end{bmatrix}$$

then, from $w \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \leq \frac{\|X\| + \|Y\|}{2}$ and $w \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = \text{Max}(w(X), w(Y))$

$$w(T) = \text{Max}(w(E), w(G), w(I)) + \frac{1}{2} \sqrt{\|BB + CC\| + \|AA + DD\|} + \frac{\|F\| + \|H\|}{2}$$

□

The next Theorem generalizes the lower bound for all diagonal operator matrices considered in the construction.

Theorem 4.3. For a special diagonal normal matrix operator of the construction and given by

$$M = \begin{bmatrix} 0 & 0 & 0 & A_1 \\ 0 & 0 & A_2 & 0 \\ 0 & \ddots & 0 & 0 \\ A_n & 0 & 0 & 0 \end{bmatrix}, \text{ the following properties hold:}$$

(i) If n is even, we have

$$w(M) \geq \sqrt[2n]{\text{Max}(w(A_i A_{n-(i+1)})^n, w(A_{n-(i+1)} T_i^n)), i = 1 \dots, n}$$

(ii) For n is odd, then

$$w(M) \geq \sqrt[2n]{\text{Max}(w(A_i A_{n-(i+1)})^n, w(A_{n-(i+1)} A_i^n, w(A_{\frac{n+1}{2}}))}, i = 1 \dots n, i \neq \frac{n+1}{2},$$

Proof.

(i) Let M be of the construction given. Suppose n is an integer, then

$$M^2 = \begin{bmatrix} A_1A_n & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_2A_{n-1} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_{\frac{n}{2}}A_{\frac{n}{2}+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & A_{\frac{n}{2}+1}A_{\frac{n}{2}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{n-1}A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & A_nA_1 \end{bmatrix}$$

which implies that

$$M^{2n} = \begin{bmatrix} (A_1A_n)^n & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & (A_2A_{n-1})^n & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & (A_{\frac{n}{2}}A_{\frac{n}{2}+1})^n & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & (A_{\frac{n}{2}+1}A_{\frac{n}{2}})^n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (A_{n-1}A_2)^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & (A_nA_1)^n \end{bmatrix}$$

For $n = 1, 2, \dots$ and so on

$$\text{Max}(w(A_iA_{n-(i+1)})^n, w(A_{n-(i+1)A_i})^n) = w(A^{2n}) \leq w^{2n}(A),$$

which is simply the inequality given by

$$w(M) \geq \sqrt[2n]{\text{Max}(w(A_iA_{n-(i+1)})^n, w(A_{n-(i+1)A_i})^n)}.$$

(ii) Next, let M be of the given construction. For an odd n , it is clear that

$$M^2 = \begin{bmatrix} A_1A_n & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & A_{\frac{n+1}{2}}A_{\frac{n+1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & A_{n-1}A_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & A_nA_1 \end{bmatrix}$$

which implies that

$$M^{2n} = \begin{bmatrix} (A_1A_n)^n & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & (A_2A_{n-1})^n & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & (A_{\frac{n+1}{2}}A_{\frac{n+1}{2}})^n & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & (A_{n-1}A_2)^n & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & (A_nA_1)^n \end{bmatrix}.$$

□

4.1 Relationship between Numerical and Spectral radii of the Normal operator matrices of our construction

Theorem 4.4. Consider $M_n(\mathbb{C}^*)$ where (\mathbb{C}^*) is an algebra, and M_n is a set of matrices of the construction in section 3, then every matrix $A_1, A_2, B_1, B_2 \in M_n(\mathbb{C}^*)$ is algebraic over (\mathbb{C}^*) . Moreover, the spectral radius γ and the numerical radius w of the operators satisfy the relationship

$$\gamma(A_1B_1 + A_2B_2) \leq \frac{1}{2}(w(B_1A_1) + w(B_2A_2)) + \frac{1}{2}\sqrt{(w(B_1A_1) - w(B_2A_2))^2 + 4 \| B_1A_2 \| \| B_2A_1 \|}$$

Proof. Intuitively

$$\begin{aligned} \gamma(A_1B_1 + A_2B_2) &= \gamma \begin{bmatrix} A_1B_1 + A_2B_2 & 0 \\ 0 & 0 \end{bmatrix} = \gamma \left(\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix} \right) \\ &= \gamma \left(\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \right) = \gamma \begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix} \leq w \begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix} \end{aligned}$$

Now, for any normal operators $A, B, C, D \in M_n$, it is well known that,

$$w \begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \frac{1}{2}(w(A) + w(D)) + \frac{1}{2}\sqrt{(w(A) - w(D))^2 + (\| B \| + \| C \|)^2}$$

and consequently, $\gamma(A_1B_1 + A_2B_2) = w \begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix}$

$$\leq \frac{1}{2}(w(B_1A_1) + w(B_2A_2)) + \sqrt{(w(B_1A_1) - w(B_2A_2))^2 + (\| B_1A_2 \| + \| B_2A_1 \|)^2}.$$

□

Corollary 4.5. Let $A, B \in M_n$ then

$$\gamma(A + B) \leq \frac{1}{2}(w(A) + w(B)) + \frac{1}{2}\sqrt{w(A) - w(B)^2 + 4\min(\| AB \|, \| BA \|)}$$

Proof. The proof follows from the proof of the previous Theorem with some few modifications □

Theorem 4.6. Let $A, B \in M_n$, then

$$\gamma(AB \pm BA) \leq w(AB) + \sqrt{\min(\| A \| \| AB^2 \|, \| B \| \| A^2B \|)}$$

and

$$\gamma(AB \pm BA) \leq w(BA) + \sqrt{\min(\| A \| \| B^2A \|, \| B \| \| BA^2 \|)}$$

Proof. Easy □

5 Positivity and Boundedness

In this section, we are interested in $B(H)_+$, the cone of the positive special normal operators on H . We consider $T = A$ where A is a normal matrix of our construction. So $A \in B(H)_+$, $\iff A = N^*N, N \in B(H)$. In fact $A \in B(H)$ if and only if $A = N^2$ for a self adjoint $N \in B(H)$. This operator N can be chosen to also be positive so that we write it as $A^{\frac{1}{2}}$ and it is unique. Positivity is related to underlying algebra to the spectral theory of A and to the operator norm $\|A\| = \sup\|Ax\|: x \in H \text{ and } \|x\| \leq 1$.

Lemma 5.1. *Let $A \in M_n(B(H))$ be a special normal matrix of our construction,*

- (i) *If $A \geq 0$, then $\|A\| = \max(\sigma(A))$.*
- (ii) *$\|A\| \leq 1$ if and only if the operator on $H \oplus H$ taking (x, y) to $(x + Ax, A^*x + y)$ is positive.*
- (iii) *A has a polar decomposition $A = U|A|$ where $|A| = (A^*A)^{\frac{1}{2}} \geq 0$.*

Proof.

(i) Let $\dim_{B(H)} H = n < \infty$ then $(\lambda_1, \dots, \lambda_n)$ define the eigenvalues of $A \in B(H)$. Now $\|A\| = \max(\lambda_i) : \lambda_i \in \sigma(A)$ are eigenvalues implying that $\|A\| = \max \sigma(A)$ for a general H which is not necessarily finite dimensional, $\sigma(A) \subseteq [0, \infty)$, so any $\alpha \in [0, \infty) \Rightarrow A \geq 0$ for if $\alpha \in [0, \infty)$ then $0 \leq \alpha < \infty$.

(ii) Since $A = A^* \Rightarrow \|A\| \leq 1 \Rightarrow -1 \leq A \leq 1$. Now

$$\begin{aligned} A : (x, y) &\rightarrow (x + Ax, A^*x + y) \\ \Rightarrow A(x, y) &= (x + Ax, A^*x + y) = (x + Ax, Ax + y)(Ax, Ay) \geq 0 \forall x, y \geq 0 \\ &\Rightarrow A \geq 0 \end{aligned}$$

(iii) Let U be unitary. This means that $U^{-1} = U^*$ or a partial isometry that is $U = UU^*U$. Now $|A| = (A^*A)^{\frac{1}{2}} |A| = (A^2)^{\frac{1}{2}} = A \geq 0$ So A satisfies partial isometry and its positivity. \square

Now, consider a matrix of the construction 3, $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ in $M_n(B(H))$.

This matrix can be viewed as an operator on H^n

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum y_{1k}x_k \\ \sum y_{2k}x_k \\ \vdots \\ \sum y_{nk}x_k \end{bmatrix}$$

If the later operator is positive then we write $[a_{ij}] \geq 0$. That is $M_n(B(H))$ has a natural cone. For $A \in C^*$ - algebra then so is M_n and thus there exists a natural cone $M_n(A)_+$. Now any map say $T : A \rightarrow B(H)$ is completely positive if and only if

$$[T(a_{ij})] \geq 0 \forall [a_{ij}] \in M_n(A)_+, n \in \mathbb{N}$$

Important in this theory is the notion of dilation. For $T : A \rightarrow B(H)$, then a dilation of T is the map $T : X \rightarrow B(K)$ for the Hilbert space K containing H , with

$$T^* = \begin{bmatrix} T(*) & * \\ * & * \end{bmatrix} : X \in A$$

Here $T : A \rightarrow B(K)$ dilates T if there is an isometry say $V : H \rightarrow K$ such that $T = V^*T'(\cdot)V$ on A

Example 5.2. Any representation on C^* algebra A that is $(* - \text{homomorphic})$; $\Pi : A \rightarrow B(H)$ is positive and completely positive. Further the amplification of π to $M_n(A)$ is positive $\forall n \in \pi$.

Theorem 5.3. A linear map $T : A \rightarrow B(H)$ is positive and hence completely positive if it can be dilated to a representation of π of A on $B(K)$ such that $B(H) \subseteq B(K)$

Proof. We need to find an inner product defined on a simple space containing H on which A has a natural algebraic representation. In this case, the space is $A \oplus H$ and we define the representation of A by:

$$\pi(a)(b \oplus x) = ab \oplus x : a, b \in A, x \in H.$$

We define the inner product on $S = A \oplus H$ by $\langle a \oplus y, b \oplus x \rangle = \langle T(b^*a)y, x \rangle$, $a, b \in A$ and $x, y \in H$. When $A = K$, then the map π is such that $\pi : A \rightarrow B(K)$ as required. \square

Definition 5.1. [2] An operator $T \in B(H)$ is called positive-normal (posinormal) if there exist a positive operator say $p \in B(H)$ called interrupter such that $TT^* = T^*PT$.

Proposition 5.1. $A \in M_n(B(H))$ is posinormal whenever there exists another $C \in M_n(B(H))$, $C \geq 0$ such that $AA^* \leq A^*CA$

Proof. It suffices to show that there exists a positive operator $C \in M_n(B(H))$ such that $AA^* \subseteq A^*CA$, then $A \geq 0$ A - normal. Let $\eta \in H$. So

$$\begin{aligned} \|A^*\eta\| &= \langle AA^*\eta, \eta \rangle^{\frac{1}{2}} \\ \|A^*\eta\|^2 &= \langle AA^*\eta, \eta \rangle \leq \langle A^*CA\eta, \eta \rangle = \langle \sqrt{C}A\eta, \sqrt{C}A\eta \rangle \leq \|\sqrt{C}\|^2 \|A\eta\|^2 \\ &\Rightarrow \|A^*\eta\| \leq \|\sqrt{C}\| \|A\eta\| \Rightarrow \|A^*\eta\| \leq \|\sqrt{C}\| \|A\eta\| : \eta \in (H) \end{aligned}$$

setting $\alpha = \|\sqrt{C}\|$. then for any $\eta \in H$ we have

$$\|A^*\eta\| \leq \alpha \|A\eta\|.$$

Thus

$$AA^* \leq \alpha A^*A : \alpha \geq 0$$

Again by the normality conditions, there exists some $B \in M_n(B(H))$ such that $A = A^*B$

$$\Rightarrow AA^* = (A^*A)(B^*A) = A^*(BB^*)A.$$

So A is a positive, normal matrix with an interrupter BB^* \square

Proposition 5.2. Let $A \in M_n(B(H))$ be positive normal matrix, then, $Ker(A) = Ker(A^2)$.

Proof. $Ker(A) \subseteq Ker(A^2)$. We need to show that $Ker(A^2) \subseteq Ker(A)$ so that $Ker(A) = Ker(A^2)$. Recall $Ker(A) = \{x : Ax = 0\}$. If $x \in Ker(A^2)$, then $A^2x = 0$. Hence $Ax \in Ker(A)$. Since $Ker(A) \subseteq Ker(A^*)$, $Ax \in Ker(A^*)$. Hence $A^*Ax = 0$. Now

$$\|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle \leq \|A^*Ax\| \|x\| = 0$$

So $Ax = 0$ and so $Ax \in Ker(A)$. $Ax \in Ker(A)$ and $A \in Ker(A^2)$

$$\Rightarrow Ker(A) = Ker(A^2).$$

\square

Theorem 5.4. Let $A \in M_n(B(H))$ be positive-normal and $B \in M_n(B(H))$ be such that $AA^* = A^*BA$ and $M \in Lat(A)$, then $A|_M$ is also a positive normal.

Proof. Let $M \in Lat(A)$. Let $\pi : A \rightarrow M$ be the dilation of H into M . then $\forall m \in M$

$$\langle (A|_M)^*m, m \rangle = \langle m, A|_M m \rangle = \langle m, Am \rangle = \langle A^*m, m \rangle = \langle A^*\pi m, m \rangle$$

Hence $(A|_M)^* = A^*\pi$ on M , $\forall m \in M$, $\|(A|_M)^*m\| = \|A^*\pi m\| = \|(\sqrt{B}A)|_M m\|$. Hence $A|_M$ is positive normal. \square

Lemma 5.5. *Let $A \in M_n(B(H))$ be completely positive normal and let $\lambda, \beta \in \sigma_p(A)$ where $\lambda \neq \beta$. If x, y are eigenvalues of λ and β respectively, then $\langle x, y \rangle = 0$*

Proof. $Ker(A - B) \subset Ker(A^* - \overline{B})$. We see that

$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle Ax, y \rangle = \langle x, A^*y \rangle = \langle x, \overline{\beta y} \rangle = \beta \langle x, y \rangle \Rightarrow \langle x, y \rangle = 0.$$

\square

6 Conclusion

This study was set up with the main objective of characterizing a special class of normal operator matrix. Motivated by the various applications of normal matrices in operator theory, engineering and information theory, the intention of this research was to provide substantial extension on the spectral theory of matrices resulting from matrix products restricted to certain Hilbert spaces. In fact, the normal operators have been used before via the subnormality and hyponormality properties to provide the proof of the non-existence of non-trivial subspaces, construction of the functional calculus and description of the spectral picture in the cyclic case for subnormal operators among others. First, we constructed a class of Normal matrices $A \in M_n$ resulting from Khatri-Rao products of matrices in direct sums of Hilbert spaces. This class of normal matrices is regarded as a special class because the blocks of the resultant matrix product formulated consists of entries of pythagorean type. So the class of consideration is taken to be the class C^1 of pythagorean triplets. The Algebraic and spectral properties concerning the Numerical radii, the spectral radii, positivity and boundedness of selected special normal matrices constructed were then studied into details, the proofs of the results provided where necessary. We also provided the relationship between the spectral radii and the numerical radii of the matrix constructed. This provides an Algebraic approach to the study of operators hence enriching the two fields. Finally, we extended the study and investigated the subnormality and hyponormality conditions of the operator matrix constructed in relation to an arbitrary operator T . We observe that our results on the normality conditions compare perfectly well with the existing results hence a beautiful extension of the study of Normal matrix operators. The results on the numerical norm, numerical radii and spectral radii are however unique and apply to the selected classes of the special normal operators. The results also enrich the study of Matrix products hence more possible applications. In particular, from the results in this paper, one can develop further operator identities/inequalities parallel to matrix results for Khatri-Rao products in an endeavour to explore more applications of this class of operator matrices in Coding Theory and Cryptography.

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Competing Interests

Authors have declared that no competing interests exist.

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