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On Fibonacci Range Labeling for Standard Shell Graphs

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

A shell $C[n,(n-3)]$ of size n is a graph obtained by taking $(n-3)$ concurrent chords in a cycle C_n on n vertices. Deb and Limaye (2002) have conjectured that all multiple shells are harmonious. The conjecture has prove to be true for uniform double shells, uniform triple shells and uniform quadruple shells. Here, we prove for a non- uniform double shells with order m and n, where $n = (m-1)$ and k-copies of a shell $C[n,(n-3)]^k$ with a union of K_2 for $n = 4$, $2K_2$ for $n = 6$ and $3K_2$ for $n = 8$, having a common end vertex joined to the apex of the shell are Fibonacci range labeling.

Keywords: Shell graph; Fibonacci range labelling; Fibonacci range graph; golden ratio.

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1 Introduction

In 1967, Rosa [1] gave a varitation on $\beta - labeling$ latter known as graceful labeling and showed that n-cycles is graceful if and only if n $\equiv$ 0 or 3 (mod 4). Frucht [2] have shown wheels $W_n = C_n + K_1$ are graceful. Delorme *et al*. [3] proved that any cycles with a chord is graceful and Koh *et al*. [4, 5], defined a cycle with a P_k-chord is graceful when $k = 3$. In particular, Deb and Limaye [6], gave harmonious labeling of cycles related graphs, such as shell graphs, cycles with maximum number of concurrent alternate chords and some families of multiple shells. Jesintha and Hilda [7, 8], proved all uniform bow graph are graceful and also proved for shell-flower graph $C[n,(n-3)]^k$ when $n = 4$ are graceful. For more survey on labeling we refer dynamic survey by Gallian [9], relevant reference and conjecture are cited in Liu and Zhang [10], Liu R Y [11] and Ma *et al*. [12]. Here, a simple finite graph G with p vertices and q edges is said to be Fibonacci range labeling if there is an injective function $f : V(G) \to \{F_2, F_3, F_4, ..., F_{p+1}\}$ such that the induced edge label are distinct and is in the form of the golden ratio. When such type of labeling exist, it is said to be Fibonacci range labeling of a graph. In this paper, we classify the class of shell graphs for k-copies of shell $C[n,(n-3)]^k$ with a union of K_2 for $n = 4$, $2K_2$ for $n = 6$ and $3K_2$ for $n = 8$ having a common end vertex joined to the apex of the shell are Fibonacci range labeling.

2 On Non–uniform Shell Graphs

Recall from Deb and Limaye [6], a shell graph is defined as a cycle C_n with $(n-3)$ chord sharing a common end point called the apex vertex and is denoted by $C[n,(n-3)]$. A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. Hence, a double shell consist of two disjoint shells with a common apex vertex. Jesintha and Hilda [9], defined a bow graph to be a double shell in which each shell has any order and shell-flower graph as k-copies of the union of shell $C[n,(n-3)]$ and K_2, where one end vertex of K_2 is joined to the apex of the shell. Further, we define graph $G = (V,E)$ is said to be a Fibonacci range graph if we label the vertices $x \in V$ with distinct labels $f(x) \to \{F_2, F_3, F_4, ..., F_{p+1}\}$ such that, when the edge $e = (\alpha, \beta)$ is labeled with $f^*(e = \alpha\beta) = \lceil \frac{f(\alpha)^2+f(\beta)^2}{f(\alpha)+f(\beta)} \rceil$ or $f^*(e = \alpha\beta) = \lfloor \frac{f(\alpha)^2+f(\beta)^2}{f(\alpha)+f(\beta)} \rfloor$, then the resulting edge gets unique label. Also the ratio of each edge to the subsequent edge is in the form of the golden ratio given by $R_t(E_{i,i+1}) = \frac{E_{i+1}}{E_i} \approx \psi$ (for larger i), where $R_t(E_{i,i+1})$ is the ratio of the resulting induced edges $(e_1, e_2), (e_2, e_3)$, ..., (e_{n-1}, e_n) and $\psi = 1.618$ known as the golden ratio. If a graph G exhibit a Fibonacci range labeling then it is defined to be a Fibonacci range graph.

Note: The Fibonacci numbers are {0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...,} here, $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, ..., but all the vertices label should be distinct, so we consider the label from F_2 only.

Proposition 2.1. *Consider α, β and l be in Z^+ with $\alpha < \beta$ then*

(i) $\alpha < \frac{\alpha^2+\beta^2}{\alpha+\beta} < \beta$

(ii) $l < \frac{l^2+(l+2)^2}{l+(l+2)} < l+2$

(iii) $l < \frac{l^2+(l+3)^2}{l+(l+3)} < l+3$

(iv) $l < \frac{l^2+(l+5)^2}{l+(l+5)} < l+5$

(v) $l-1 < \frac{1^2+l^2}{1+l} < l$

Lemma 2.1. *The generating function for Fibonacci labeling with $\alpha\beta$ are, respectively:*

$$\phi(x) = 1 + x\phi + x^2\phi$$

and

$$s(x) = \frac{\phi(x)}{x}$$

Corollary 2.2. *For any graph G with corresponding vertices label as a series of Fibonacci number* $\sum_{n=2}^{\infty} F_n$ *then the edge gets labeled between the values* $L_2, L_3, L_4, ..., L_n$ *where* L_n *is the Lucas number and the relation between the vertices and edges gives*

$$v_i, v_{i+1} \rightarrow e_j = \begin{cases} 1,2 \ \rightarrow \ 1 & for \ i=1, \ j=1 \\ 2,3 \ \rightarrow \ 3 & for \ i=2, \ j=2 \\ 3,5 \ \rightarrow \ 4 & for \ i=3, \ j=3 \\ 5,8 \ \rightarrow \ 7 & for \ i=4, \ j=4 \\ 8,13 \ \rightarrow \ 11 & for \ i=5, \ j=5 \\ \dots & \\ F_{n-1}, F_n \ \rightarrow \ E_n & for \ i=n, \ j=n \end{cases} \tag{2.1}$$

The above generate the relation $E_n \approx [\ \psi^n + \frac{1}{2}\ ]$ *which is similar to the Lucas number identities.*

Theorem 2.3. *The graph* $C[n,(n-3) \cup K_2]^k$*, for* $n = 4$*, where one end vertex of* K_2 *is joined to the apex are a Fibonacci range labeling.*

Proof. Let $C[n,(n-3) \cup K_2]^k$, where $n = 4$ be the shell graph with union of K_2 where one end vertex is joined to the apex of the shell (see Fig. 3). The order of the graph (order exclude the apex) is $p = k(n-1) + k$ and size $q = k(n+2)$. Define the labeling function f, we label the vertices of the graph as follows, $f(u_0) = 1$ and $f(v_i) = f_{i+2}$, where $1 \leq i \leq n$. From the above vertex labeling operation the edge gets label as

(*i*) $f(u_0v_i) = f(v_i) - 1$, for $i = 1$
$f(u_0v_i) = f(v_i)$, otherwise

For 1^{st} k-copies

(*i*) $f(v_1v_2)$, $f(v_2v_3) = 2, 4$

For 2^{nd} k-copies

(*i*) $f(v_iv_{i+1}) = f(v_i) + F_i$, for $i = 5, 6$

For m^{th} k-copies

(*i*) $f(v_{n-2}v_{n-1}) = f(v_{n-2}) + F_{n-2}$, for $m = (n-2), (n-1)$

From the above computations, by observation the edge values are distinct. Hence, we conclude that the graph $C[n,(n-3) \cup K_2]^k$, for $n = 4$ is a Fibonacci range labeling. The calculated ratio of the induced edge values for $C[4,(4-3) \cup K_2]^3$, here $k = 3$

$R_t(e_ie_{i+1}) = 3, 1.666, 1.625, 1.615, 1.619, 1.618, ...,$

$R_t(e_i^1e_{i+1}^1) = 2.00$, $R_t(e_j^2e_{j+1}^2) = 1.611$, $R_t(e_k^3e_{k+1}^3) = 1.618$

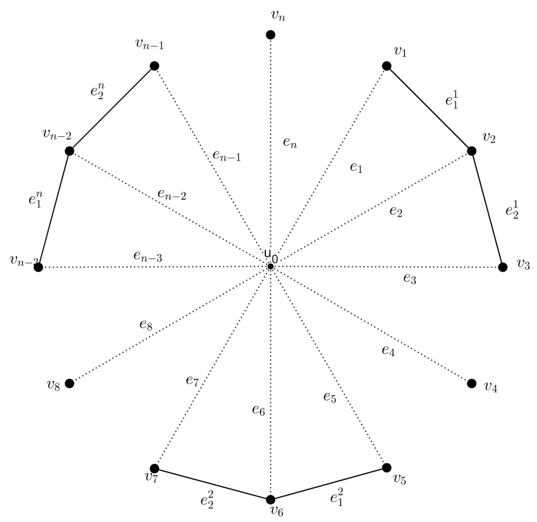

Fig. 1. $C[n,(n-3)\cup K_2]^k$**, for** $n = 4$

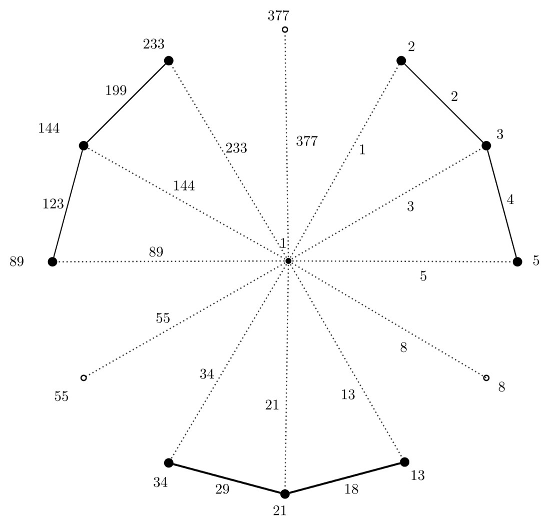

Fig. 2. Fibonacci range graph for $C[4,(4-1)\cup K_2]^3$

We observe from the calculated values converges to $\approx 1.618 = \psi$ but restricted only for larger $(i, j, k, ...,)$. Hence, the graph $C[n,(n-3)\cup K_2]^k$, for $n = 4$ with one end vertex of K_2 joined to the apex converges to ψ only when taken for higher order of k-copies of the shell.

□

Theorem 2.4. *The graph $C[n,(n-3)\cup 2K_2]^k$ for $n = 6$, where two end vertex of $2K_2$ is joined to the apex are Fibonacci range labeling.*

Proof. Let $C[n,(n-3)\cup 2k_2]^k$ where $n = 6$ be the shell graph with union of $2K_2$ end vertex joined to the apex of the shell. The order of the graph (order exclude the apex) is $p = kn + k$ and size $q = k(2n-2) + k$. Define the labeling function f then we label the vertices of the graph when $n = 6$ as follows, $f(u_0) = 1$ and $f(v_i) = f_{i+2}$, where $1 \leq i \leq n$ from the above vertex labeling operation the edge get labels as

(*i*) $f(u_0v_i) = f(v_i) - 1$, for $i = 1$
$f(u_0v_i) = f(v_i)$, otherwise

For 1^{st} k-copies

(*ii*) $f(v_1v_2) = 2$

(*iii*) $f(v_iv_{i+1}) = f(v_i) + F_i$, for $i = 2, 3, 4$

For 2^{nd} k-copies

(*iv*) $f(v_jv_{j+1}) = f(v_j) + F_j$, for $j = 8, 9, 10, 11$

For 3^{th} k-copies

(*v*) $f(v_kv_{k+1}) = f(v_k) + F_k$, for $k = 15, 16, 17, 18$

For m^{th} k-copies

(*vi*) $f(v_{n-3}v_{n-2}) = f(v_{n-3}) + F_{n-3}$, for $m = (n-6), (n-5), (n-4), (n-3)$

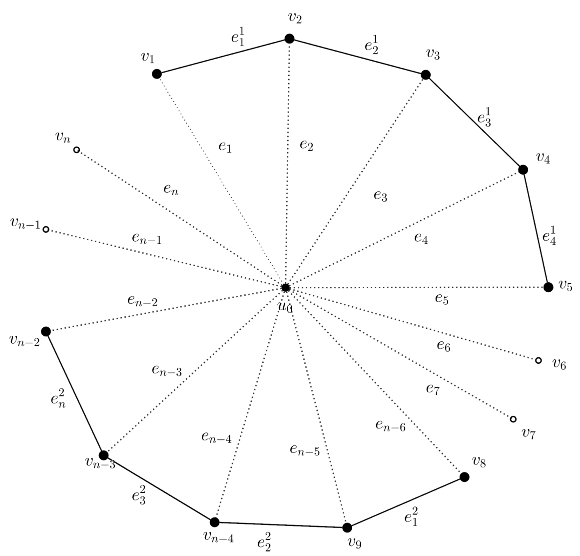

Fig. 3. Generalised graph $C[n,(n-3)\cup 2K_2]^k$ for $n = 6$

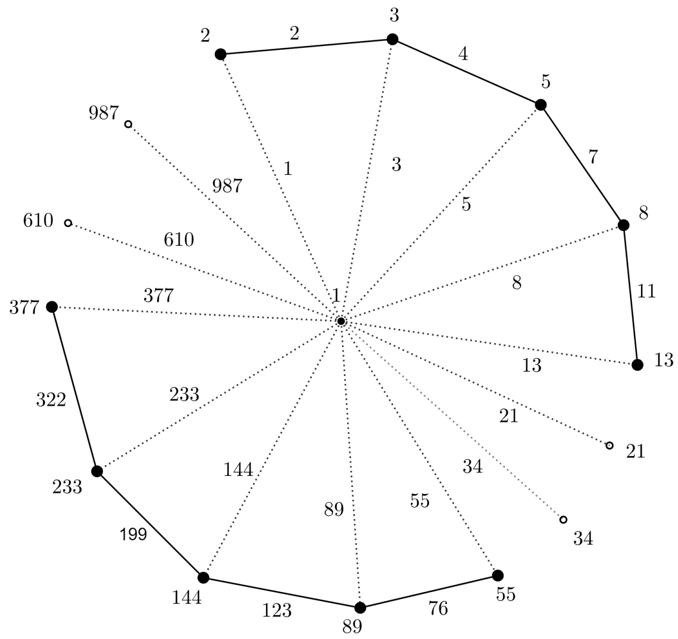

Fig. 4. Fibonacci range graph for $C[6,(6-3)\cup 2K_2]^2$

By observation the edge values are distinct by the above operation. The calculated ratio of the induced edge values for $C[6,(6-3)\cup 2K_2]^2$ here $k=2$

$R_t(e_ie_{i+1}) = 3$, 1.666, 1.6, 1.625, 1.615, 1.619, 1.618, 1.618, ...,

$R_t(e_i^1e_{i+1}^1) = 2$, 1.75, 1.60

$R_t(e_j^2e_{j+1}^2) = 1.618$, 1.618, 1.618

As in the case of the calculated values it converges to $\approx \psi$ but only when taken for higher order of $(i,j,...,)$. Therefore, the graph $C[n,(n-3)\cup 2K_2]^k$ for $n=6$, with two end vertex of $2K_2$ joined to the apex converges to ψ only when taken for higher order of k-copies of the shell. □

Theorem 2.5. *The graph $C[n,(n-3)\cup 3K_2]^k$, for $n=8$ of a Shell graph where three end vertex of K_2 is joined to the apex vertex are Fibonacci range labeling.*

Proof. Let $C[n,(n-3)\cup K_2]^k$ where $n=8$ be the shell graph with union of $3K_2$ end vertex joined to the apex of the shell. The order of the graph (order exclude the apex) is $p=kn+2k$ and size $q=k(2n-2)+2k$. Define a function f then we label the vertices of the shell graph when $n=8$ as follows, $f(u_0)=1$ and $f(v_i)=f_{i+2}$, where $1\le i\le n$ from the above vertex labeling operation the edge gets label as

(*i*) $f(u_0v_i) = f(v_i) - 1$, for $i=1$
$f(u_0v_i) = f(v_i)$, otherwise

For 1^{st} k-copies

(ii) $f(v_1v_2) = 2$

(iii) $f(v_iv_{i+1}) = f(v_i) + F_i$, for $i = 2, 3, 4, 5, 6$

For 2^{nd} k-copies

(iv) $f(v_jv_{j+1}) = f(v_j) + F_j$, for $j = 11, 12, 13, 14, 15, 16$

For 3^{th} k-copies

(v) $f(v_kv_{k+1}) = f(v_k) + F_k$, for $k = 21, 22, 23, 24, 25, 26$

For m^{th} k-copies

(vi) $f(v_{n-4}v_{n-3}) = f(v_{n-4}) + F_{n-4}$, for $m = (n-9), (n-8), (n-7), (n-6), (n-5), (n-4)$

From the above computations, the edge values are distinct by the above operation.

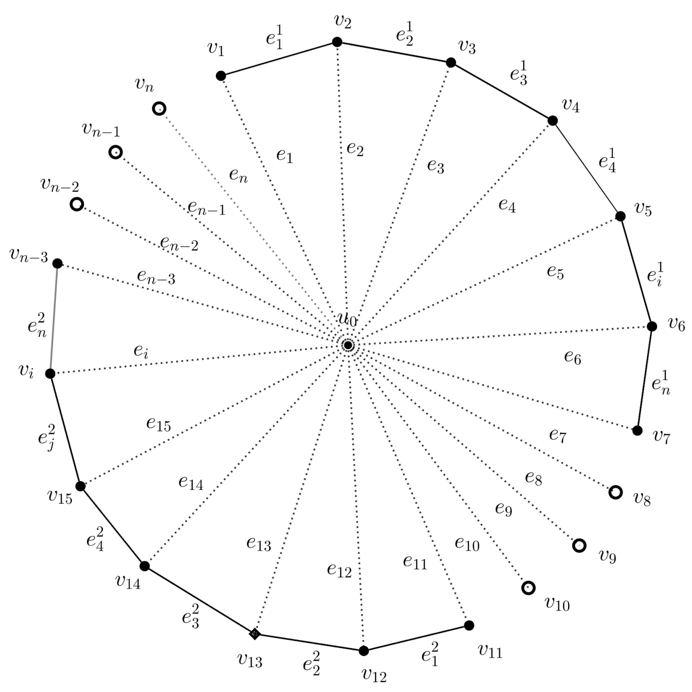

Fig. 5. $C[n,(n-3)\cup 3K_2]^k$ **for** $n = 8$

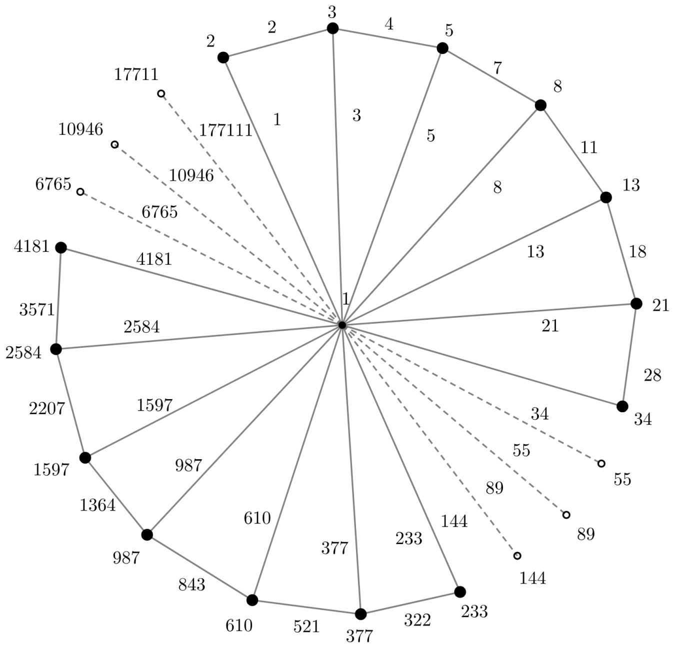

Fig. 6. Fibonacci range graph for $C[8,(8-3)\cup 3K_2]^2$

The calculated ratio of the induced edge values for $C[8,(8-3)\cup K_2]^2$, here $k=2$

$R_t(e_ie_{i+1})$ = 3, 1.666, 1.6, 1.625, 1.615, ...,

$R_t(e_i^1e_{i+1}^1)$ = 2, 1.75, 1.60, 1.636, 1.611, ...,

$R_t(e_j^2e_{j+1}^2)$ = 1.618, 1.618, 1.618, 1.618, ...,

Therefore, the graph $C[n,(n-3)\cup 3K_2]^k$ for $n=8$, with three end vertex of $3K_2$ joined to the apex converges to ψ when taken for higher order of k-copies of the shell.

□

3 Conclusion

In this article, we presented a brief result related to Deb and Limaye [6] conjecture that all multiple shells are harmonious. We generalised the class of shell graphs and have proved that multiple shells with non-uniform order and shell graph with union of K_2 for $n=4$, $2K_2$ for $n=6$, $3K_2$ for $n=8$ are Fibonacci range graphs. We hope that the generalised result leads to better understanding of the conjecture of Deb and Limaye on multiple shells which yields a distinctive result on such class of graphs with the framing of Fibonacci range labeling.

Competing Interests

Authors have declared that no competing interests exist.

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