Hindawi
Abstract and Applied Analysis
Volume 2022, Article ID 9558928, 18 pages
https://doi.org/10.1155/2022/9558928

Research Article

Measure (ω, c)-Pseudo-Almost Periodic Functions and Lasota-Wazewska Model with Ergodic and Unbounded Oscillating Oxygen Demand

James Larrouy [1] and Gaston M. N'Guérékata [2]

[1] *Department of Mathematics, Université des Antilles, Pointe-à-Pitre, Guadeloupe, France*
[2] *NEERLab, Department of Mathematics, Morgan State University, Baltimore, MD, USA*

Correspondence should be addressed to Gaston M. N'Guérékata; gaston.nguerekata@morgan.edu

Received 2 May 2022; Accepted 28 June 2022; Published 26 July 2022

Academic Editor: Nasser-Eddine Tatar

The primary aim of this work is to introduce a new class of functions called μ-(ω, c)-pseudo-almost periodic functions. Using the measure theory, we generalize in a natural way some recent works and study some properties of those μ-(ω, c)-pseudo-almost periodic functions including two new composition results which play a crucial role for the existence of some μ-(ω, c)-pseudo-almost periodic solutions of certain semilinear differential equations and partial differential equations. We also investigate the existence and uniqueness of the μ-(ω, c)-pseudo-almost periodic solutions for some models of Lasota-Wazewska equation with measure (ω, c)-pseudo-almost periodic coefficient and mixed delays.

1. Introduction

Most of the natural phenomena we consider as periodic are in fact almost periodic; in other words, they are periodic up to epsilon. The concept of almost periodic functions was introduced in the literature in the mid-1920s by the Danish mathematician Harald Bohr [1]. It was later generalized in various directions by many researchers [2–12]. As we all know, many phenomena in nature have oscillatory character, and their mathematical models have led to the introduction of certain classes of functions to describe them. Such a class form pseudo-almost periodic functions which is a natural generalization of the concept of almost periodicity (in Bohr's sense). In this work, we introduce the notion of measure (ω, c)-pseudo-almost periodic functions (or μ-(ω, c)-pseudo-almost periodic functions) with values in a complex Banach space and enlighten their applications throughout the study of a biological model. This work generalizes the concept of μ-pseudo-almost periodic functions introduced by Blot et al. [4] which already generalizes the class of weighted pseudo-almost periodic functions of Diagana [6, 13]. Here, we investigate many interesting properties of this new class of functions and present new and more general results based on measure theory that extend the existing ones.

The concept of (ω, c)-periodicity was introduced by Alvarez et al. [2] motivated by the qualitative properties of solutions to the Mathieu linear second-order differential equation

$$y''(t) + [a - 2q \cos(2t)]y(t) = 0, \tag{1}$$

arising in seasonally forced population dynamics. Further on, Alvarez et al. proposed a new concept of (ω, c)-pseudoperiodicity and proved the existence of positive (ω, c)-pseudo-periodic solutions to the Lasota-Wazewska equation with (ω, c)-pseudoperiodic coefficients

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, t \geq 0. \tag{2}$$

This equation describes the survival of red blood cells in the blood of an animal. The works of Khalladi et al. [14]

have shown that (ω, c)-pseudoperiodic functions can be also solutions time varying impulsive differential equations and linear delayed equations.

The concept of pseudo-almost periodicity was introduced in the literature in the early nineties by Zhang [11, 12, 15], as a natural generalization of the classical almost periodicity in the sense of Bohr. Then, Diagana [6, 13] introduced the concept of weighted pseudo-almost periodicity which generalizes the latter, and the author gave some properties of the space of weighted pseudo-almost periodic functions such as the completeness and a composition theorem. The concept of weighted pseudo-almost periodic functions became an interesting field of dynamical systems that attracted many authors. A few years later, Blot et al. [4] came up with a new concept of weighted pseudo-almost periodic functions under the light of measure theory. Giving a positive measure μ on $\mathbb{R}$, they defined the concept of μ-pseudo-almost periodic functions as follows: it is said that a function f is μ pseudo-almost periodic if

$$f = g + \varphi, \tag{3}$$

where g is almost periodic and φ is μ-ergodic in the sense that

$$\lim_{r \longrightarrow \infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|\varphi(t)\| d\mu(t) = 0. \tag{4}$$

Here, the classical theory of weighted pseudo-almost periodicity became a particular case of Blot et al. approach. Indeed, one can observe that a weighted pseudo-almost periodic function of weight ρ is μ-pseudo-almost periodic where the measure μ is absolutely continuous with respect to the Lebesgue measure, and its Radon-Nikodym derivative is ρ:

$$d\mu(t) = \rho(t)dt. \tag{5}$$

In their work, Blot et al. have investigated many important results on the theory of μ-pseudo-almost periodicity; they studied the completeness and provided a composition theorem on the functional space of μ-pseudo-almost periodic functions. They also gave some applications for evolution equations which include reaction-diffusion systems and partial differential equations.

In this work, we introduce a new class of μ-(ω, c)-ergodic components, and we investigate many important results on the new theory of μ-(ω, c)-pseudo-almost periodic functions. We study the completeness and the composition theorem on the functional space of μ-(ω, c)-pseudo-almost periodic functions.

The organization of this work is as follows: in the next section, we recall the basic definitions and properties of μ-pseudo-almost periodic functions. In Section 3, we give the new concept of μ-(ω, c)-pseudo-almost periodicity and study the convolution product on the spaces of c-bounded functions, μ-(ω, c)-ergodic functions, and μ-(ω, c)-pseudo-almost periodic functions. In Section 4, we introduce the concept of (ω, c)-type compactness, and then we study a composition theorem which plays a crucial role to study the existence of μ-(ω, c)-pseudo-almost periodic solution for a perturbed semilinear system. In Section 5, we propose a more realistic Lasota-Wazewska model than the existing ones due to (ω, c)-periodicity, and then we study the existence and uniqueness of μ-(ω, c)-pseudo-almost periodic solutions for the model, using the completeness and composition results.

2. Terminology and Definitions

In this section, we review a few notations, definitions, and lemmas which will be utilized throughout this paper.

Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ be complex Banach spaces. Throughout this work, $\mathrm{C}(\mathbb{R}, \mathbb{X})$ and $\mathrm{BC}(\mathbb{R}, \mathbb{X})$ (respectively, $\mathrm{C}(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $\mathrm{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the Banach spaces consisting of all continuous functions and all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$ (respectively, from $\mathbb{R} \times \mathbb{Y}$ to $\mathbb{X}$) equipped with the supremum norm

$$\|f\|_\infty = \sup_{t \in \mathbb{R}} \|f\|. \tag{6}$$

Let us first recall the notion of (ω, c)-periodicity.

Definition 1. (see [2]). Let $\omega > 0$ and $\mathrm{c} \in \mathbb{C} \setminus \{0\}$. A function $\mathrm{f} \in \mathrm{C}(\mathbb{R}, \mathbb{X})$ is said to be (ω, c)-periodic if

$$f(t + \omega) = cf(t), \text{for each } t \in \mathbb{R}. \tag{7}$$

In this case, ω is called a c-period of the function f.

We denote by $\mathrm{P}_{(\omega,c)}(\mathbb{R}, \mathbb{X})$ the vector space of all (ω, c)-periodic functions from $\mathbb{R}$ to $\mathbb{X}$. One can note that the space $\mathrm{P}_{(\omega,c)}(\mathbb{R}, \mathbb{X})$ contains the spaces of periodic, antiperiodic, and Bloch periodic functions among others (respectively, taking $c = 1$, $c = -1$, and $c = e^{\mathrm{ikt}}$) (see [16] for more details).

Proposition 2. (see [2]). *Let $f \in \mathrm{C}(\mathbb{R}, \mathbb{X})$. Then, $f \in P_{(\omega,c)}(\mathbb{R}, \mathbb{X})$ if and only if*

$$f(t) = c^{t/\omega} u(t), u(t) \in P_{\omega,1}(\mathbb{R}, \mathbb{X}). \tag{8}$$

Using the principal branch of the complex Logarithm, $c^{t/\omega}$ is defined as

$$c^{t/\omega} := \exp\left(\frac{t}{\omega} Log(c)\right) = c^\wedge(t), \tag{9}$$

and we will use the notation $|c|^\wedge(t) := |c^\wedge(t)| = |c|^{t/\omega}$.

Now, we recall some properties of almost periodic and (ω, c)-pseudo-almost periodic functions.

Definition 3. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$, there exists $l > 0$, such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l]$ with

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| < \varepsilon. \tag{10}$$

The vector space consisting of all (Bohr) almost periodic functions is denoted by $AP(\mathbb{X})$.

It is well known that a continuous function $f : \mathbb{R} \longrightarrow \mathbb{X}$ is almost periodic if and only if the set

$$\{f_\tau : \tau \in \mathbb{R}\} \tag{11}$$

is relatively compact in $BC(\mathbb{R}, \mathbb{X})$, where the function f_τ is defined by

$$f_\tau(t) = f(t+\tau), t \in \mathbb{R}. \tag{12}$$

Such number τ in (10) is called ε-translation number of $f(t)$, and we denote by $\mathcal{T}_\varepsilon(f)$ the set of all ε-translation numbers of f. This set has the following property:

Given any $f \in AP(\mathbb{X})$,

(1) If $\tau \in \mathcal{T}_\varepsilon(f)$, then $-\tau \in \mathcal{T}_\varepsilon(f)$.

This concept has been extended by Khalladi et al. [17] as follows:

Definition 4. (see [17]). A function $f \in C(\mathbb{R}, \mathbb{X})$ is called (ω, c)-almost periodic if and only if the function

$$f_{(\omega,c)}(t) := c^{-t/\omega} f(t), t \in \mathbb{R} \tag{13}$$

belongs to $AP(\mathbb{X})$.

The vector space consisting of all (ω, c)-almost periodic functions is denoted by $AP_{\omega,c}(\mathbb{X})$.

Unless specified otherwise, in the remainder of the paper, we will always assume that $c \in \mathbb{C} \setminus \{0\}$ and $\omega \in \mathbb{R}_+^\star \cap \mathcal{T}_\varepsilon(f)$. Furthermore, the principal branches are always used for taking powers of complex numbers.

In the following, we will keep the notation: $f_{(\omega,c)}(t) := c^{-t/w} f(t)$.

Remark 5. When $c = 1$, $AP_{\omega,c}(\mathbb{X}) = AP(\mathbb{X})$.

Remark 6. One can note that in our paper, contrary to the paper [17], ω is not only positive but it belongs also to the set of all ε-translation number of f. This condition yields $AP_{\omega,1}(\mathbb{X}) := AP_\omega(\mathbb{X}) = AP(\mathbb{X})$.

In order to conserve the periodic structure of (ω, c)-periodic type functions, we need to use an (ω, c)-norm which can be defined as

$$\|f\|_{(\omega,c)} := \sup_{t\in\mathbb{R}} \left\|c^\wedge(-t) f(t)\right\|. \tag{14}$$

(ω, c)-norms were introduced in the literature by Alvarez et al. taking the supremum norm not on the whole $\mathbb{R}$ but on the principal c-period interval $[0, \omega]$ of the (ω, c)-periodic considered function in order to handle the (ω, c)-periodicity properties of f (see in [2, 3, 16] for more details). We have the following completeness result.

Remark 7. We say that f is c-bounded when $\|f\|_{(\omega,c)} < \infty$.

Proposition 8. (see [14]). *$(AP_{\omega,c}(\mathbb{X}), \|\cdot\|_{(\omega,c)})$ is a Banach space.*

Proposition 9. (see [18]). *$AP_{\omega,c}(\mathbb{X})$ is translation invariant and closed under the multiplication with complex scalars.*

Now, we recall the concept of μ-pseudo-almost periodic functions introduced by Blot et al. [4].

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbb{R} (a \leq b)$.

Definition 10. (see [4]). Let $\mu \in \mathcal{M}$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be μ-ergodic if

$$\lim_{r\longrightarrow\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0. \tag{15}$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 11. (see [4]). Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be μ-pseudo-almost periodic if f is written in the form

$$f = g + \varphi, \tag{16}$$

where $g \in AP(\mathbb{X})$ and $\varphi \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$.

We denote the space of all such functions by $PAP(\mathbb{R}, \mathbb{X}, \mu)$.

Proposition 12. (see [4]). *Let $\mu \in \mathcal{M}$. Then, $(\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

In the last section of this work, the following result will be required.

Lemma 13. *If $f, g \in PAP(\mathbb{R}, \mathbb{C}, \mu)$, then $fg \in PAP(\mathbb{R}, \mathbb{C}, \mu)$.*

Proof. Since $f, g \in PAP(\mathbb{R}, \mathbb{C}, \mu)$, then they have following decompositions $f = f_1 + f_2$ and $g = g_1 + g_2$ where $f_1, g_1 \in AP(\mathbb{C})$ and $f_2, g_2 \in \mathcal{E}(\mathbb{R}, \mathbb{C}, \mu)$. Then, we have

$$fg = f_1 g_1 + f_1 g_2 + g_1 f_2 + f_2 g_2. \tag{17}$$

First, we show that the product $f_1 g_1 \in AP(\mathbb{C})$. If we take

$f_1 = g_1$, we have

$$\left\| (f_1)^2(t+\tau) - (f_1)^2(t) \right\| \| = \| (f_1)(t+\tau) + (f_1)(t) \| \times \| (f_1)(t+\tau) - (f_1)(t) \|. \tag{18}$$

It can be easily seen that since f_1 is bounded, then there exists $M \in \mathbb{R}_+$ such that

$$\|f_1\| \le M. \tag{19}$$

Thus, it comes the following

$$\left\| (f_1)^2(t+\tau) - (f_1)^2(t) \right\| \le 2M\varepsilon \le \varepsilon'. \tag{20}$$

Then, $(f_1)^2 \in AP(\mathbb{C})$. Now, one can note that $f_1 g_1 = 1/4((f_1+g_1)^2 - (f_1-g_1)^2)$. Since $(f_1+g_1)^2 \in AP(\mathbb{C})$ and $(f_1-g_1)^2 \in AP(\mathbb{C})$, then $f_1 g_1 \in AP(\mathbb{C})$.

Now, for $(f_1 g_2 + g_1 f_2 + f_2 g_2)$ one has that

$$\begin{aligned} &\frac{1}{\mu([-r,r])} \int_{-r}^{r} (|((f_1)(g_2) + (g_1)(f_2) + (f_2)(g_2))(t)|) d\mu(t) \\ &\le \frac{1}{\mu([-r,r])} \int_{-r}^{r} \left(\|f_1\|_\infty |(g_2)(t)| + \|g_1\|_\infty |(f_2)(t)| \right. \\ &\quad \left. + \|f_2\|_\infty |(g_2)(t)| \right) d\mu(t). \end{aligned} \tag{21}$$

And consequently, since $f_2, g_2 \in \mathscr{E}(\mathbb{R}, \mathbb{C}, \mu)$, we have

$$\lim_{r \longrightarrow +\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} (|((f_1)(g_2) + (g_1)(f_2) + (f_2)(g_2))(t)|) d\mu(t) = 0. \tag{22}$$

The proof is complete. □

We end this section recalling the following lemma due to Schwartz [19].

Lemma 14. *If $\Psi \in C(\mathbb{X}, \mathbb{Y})$, then for each compact set $\mathscr{K}$ in $\mathbb{X}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_1, x_2 \in \mathbb{X}$, one has*

$$x_1 \in \mathscr{K} \text{ and } \|x_1 - x_2\| \le \delta \Rightarrow \|\Psi(x_1) - \Psi(x_2)\| \le \varepsilon. \tag{23}$$

3. Measure (ω, c)-Pseudo-Almost Periodic Functions

In this section, we introduce the new concepts of μ-(ω, c)-ergodic functions and the μ-(ω, c)-pseudo-almost periodic functions. The notion of μ-(ω, c)-pseudo-almost periodic functions is a generalization of μ-pseudo-almost periodic functions introduced by Blot et al. [4] which now becomes the particular case $c = 1$ of our work. It is also a generalization of the concept of weighted pseudo-almost periodicity given by Diagana [6, 13] and consequently, this work generalizes that of Zhang [11, 12, 15] on the classical pseudo-almost periodicity.

Here, we introduce the space $BC(\mathbb{R}, \mathbb{X}, c)$ (resp.,$BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c)$), where $BC(\mathbb{R}, \mathbb{X}, c)$ (resp., $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c)$) denotes the Banach space consisting of all c-bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$ (resp., from $\mathbb{R} \times \mathbb{Y}$ to X) equipped with the (ω, c)-norm $\|\cdot\|_{(\omega,c)}$ defined in Section 2.

Remark 15. One can note that in the case $c = 1$,

$$\begin{aligned} \left(BC(\mathbb{R}, \mathbb{X}, c), \|\cdot\|_{(\omega,c)} \right) &= \left(BC(\mathbb{R}, \mathbb{X}), \|\cdot\|_\infty \right), \\ \left(BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c), \|\cdot\|_{(\omega,c)} \right) &= \left(BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X}), \|\cdot\|_\infty \right). \end{aligned} \tag{24}$$

Moreover, we have the following result.

Theorem 16. *Let $c, c' \in \mathbb{C} \setminus \{0\}$ and $f \in BC(\mathbb{R}, \mathbb{X}, c)$. Then, $f \in BC(\mathbb{R}, \mathbb{X}, c')$ is and only if $|c| = |c'|$.*

Remark 17. It can be easily seen that when $c \ne 1$, the space $BC(\mathbb{R}, \mathbb{X}, c)$ does not contain the space of constant functions.

We begin this part with the following helpful convolution theorem for c-bounded functions.

Let $\mathscr{L}(\mathbb{X})$ be the space of bounded linear maps from the complex Banach space $\mathbb{X}$ into itself. We denote $L^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$ the Lebesgue space with respect to the Lebesgue measure on $\mathbb{R}$.

Remark 18. One can note that if $h \in L^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$, then $\phi(\cdot) \coloneqq c^\wedge(\cdot) h(\cdot) \in L^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$ but $\widetilde{\phi} \coloneqq (c^\wedge(-\cdot)\phi(\cdot)) \in L^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$.

Theorem 19. *Let $f \in BC(\mathbb{R}, \mathbb{X}, c)$ and $\widetilde{\phi} \coloneqq (c^\wedge(-\cdot)\phi(\cdot)) \in L^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$, and then the convolution product of $f å\phi$ defined by*

$$(f \star \phi)(t) = \int_{-\infty}^{+\infty} \phi(s) f(t-s) ds, \text{for } t \in \mathbb{R} \tag{25}$$

is c-bounded.

Proof. Let $f \in BC(\mathbb{R}, \mathbb{X}, c)$. In order to state that $(f \star \phi) \in BC(\mathbb{R}, \mathbb{X}, c)$, we consider the function $\rho_n : \mathbb{R} \longrightarrow \mathbb{X}$ $(n \in N, n \ge 1)$ defined by

$$\rho_n(t) = \int_{-n}^{n} \phi(s) f(t-s) ds. \tag{26}$$

Observing that

$$c^\wedge(-t)\rho_n(t) = \int_{-n}^{n} c^\wedge(-s)\phi(s) c^\wedge(-t+s) f(t-s) ds. \tag{27}$$

it is clear that ρ_n is c-bounded on $\mathbb{R}$. We deduce that

$\rho_n(t)$ is continuous by using the uniform continuity of f on all compact subsets of $\mathbb{R}$. Consequently, $\rho_n(t) \in BC(\mathbb{R}, \mathbb{X}, c)$, which means that $c^\wedge(-t)\rho_n(t) \in BC(\mathbb{R}, \mathbb{X})$, and from the following inequality:

$$\begin{aligned}&\left\|c^\wedge(-t)((f\star\phi)(t)-\rho_n(t))\right\|\\&\quad\le \|f\|_{(\omega,c)}\left(\int_{-\infty}^{-n}\left\|c^\wedge(-s)\phi(s)\right\|ds+\int_{n}^{+\infty}\left\|c^\wedge(-s)\phi(s)\right\|ds\right).\end{aligned} \tag{28}$$

We deduce that $\lim_{n\longrightarrow+\infty}c^\wedge(-t)\rho_n(t) = c^\wedge(-t)(f\star\phi)(t)$ uniformly on $\mathbb{R}$. Therefore, $f\star\phi \in BC(\mathbb{R}, \mathbb{X}, c)$. □

3.1. On μ-(ω, c)-Ergodicity

First, we introduce the new concept of μ-(ω, c)-ergodic functions.

Definition 20. Let $\mu \in \mathscr{M}$. A function $f \in BC(\mathbb{R}, \mathbb{X}, c)$ is said to be μ-(ω, c)-ergodic if

$$\lim_{r\longrightarrow\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)=0. \tag{29}$$

We denote the space of all such functions by $\mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$.

We establish a completeness result for μ-(ω, c)-ergodic functions.

Proposition 21. *Let $\mu \in \mathscr{M}$. Then, $(\mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_{(\omega,c)})$ is a Banach space.*

Proof. It is clear that $\mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a vector subspace of $BC(\mathbb{R}, \mathbb{X}, c)$. We show that $\mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is closed in $BC(\mathbb{R}, \mathbb{X}, c)$. Let $(f_n) \subset \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ be a Cauchy sequence converging to f uniformly in $\mathbb{R}$. From $\mu(\mathbb{R}) = +\infty$, it follows that $\mu([-r, r]) > 0$ for r sufficiently large. We have that

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)&\le \|f-f_n\|_{(\omega,c)}\\&\quad+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left\|(f_n)_{(\omega,c)}(t)\right\|d\mu(t).\end{aligned} \tag{30}$$

Then,

$$\begin{aligned}&\limsup_{r\longrightarrow\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)\\&\quad\le \|f-f_n\|_{(\omega,c)}, \text{ for all } n\in\mathbb{N}.\end{aligned} \tag{31}$$

Since $\lim_{n\longrightarrow+\infty}\|f-f_n\|_{(\omega,c)} = 0$, we deduce that

$$\lim_{r\longrightarrow\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)=0. \tag{32}$$

□

Now, we characterize the space of μ-(ω, c)-ergodic functions with the following theorem.

Theorem 22. *Let $\mu \in \mathscr{M}$ and I be an interval such that $\mu(I) < \infty$.*

Let $f \in BC(\mathbb{R}, \mathbb{X}, c)$, and then following assertions are equivalent:

(1) $f \in \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$,

(2) $\lim_{r\longrightarrow\infty} 1/\mu([-r, r] \setminus I)\int_{[-r,r]\setminus I}\|f_{(\omega,c)}(t)\|d\mu(t) = 0$,

(3) For any $\varepsilon > 0$, $\mu(\{t \in [-r, r] \setminus I : \|f_{(\omega,c)}(t)\| > \varepsilon\})/\mu([-r, r] \setminus I) = 0$.

Proof. First, note that since $f \in BC(\mathbb{R}, \mathbb{X}, c)$, then $\Lambda \coloneqq \int_I\|f_{(\omega,c)}\|d\mu(t) < \infty$. Setting $\mathscr{I}_r = [-r, r]$ for any $r > 0$ and taking r such that $I \subset \mathscr{I}_r$ and $\mu(\mathscr{I}_r \setminus I) > 0$, it comes that

$$\begin{aligned}&\frac{1}{\mu(\mathscr{I}_r\setminus I)}\int_{\mathscr{I}_r\setminus I}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)\\&\quad=\frac{\mu(\mathscr{I}_r)}{\mu(\mathscr{I}_r)-\mu(I)}\left(\frac{1}{\mu(\mathscr{I}_r)}\int_{\mathscr{I}_r}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)-\frac{\Lambda}{\mu(\mathscr{I}_r)}\right).\end{aligned} \tag{33}$$

Since $\mu(\mathbb{R}) = \infty$, we deduce that assertions 3.6 and 3.6 are equivalent.

Now, we set the following:

$$\begin{aligned}\Gamma_r^\varepsilon &\coloneqq \left\{t\in\mathscr{I}_r\setminus I : \left\|f_{(\omega,c)}(t)\right\|>\varepsilon\right\} \text{ and } \Lambda_r^\varepsilon\\&\coloneqq \left\{t\in\mathscr{I}_r\setminus I : \left\|f_{(\omega,c)}(t)\right\|\le\varepsilon\right\}.\end{aligned} \tag{34}$$

If 3 holds, from the following equality

$$\int_{\mathscr{I}_r}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)=\int_{\Gamma_r^\varepsilon}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)+\int_{\Lambda_r^\varepsilon}\left\|f_{(\omega,c)}(t)\right\|d\mu(t), \tag{35}$$

we deduce for r large enough that

$$\frac{1}{\mu(\mathscr{I}_r\setminus I)}\int_{\mathscr{I}_r\setminus I}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)\le \|f\|_{(\omega,c)}\frac{\mu(\Gamma_r^\varepsilon)}{\mu(\mathscr{I}_r\setminus I)}+\varepsilon. \tag{36}$$

Then, from previous inequality, we have that for all $\varepsilon > 0$,

$$\lim_{r\longrightarrow\infty}\frac{1}{\mu(\mathscr{I}_r\setminus I)}\int_{\mathscr{I}_r\setminus I}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)\le\varepsilon, \tag{37}$$

and consequently, assertion 2 holds.

The last implication is deduced using the following inequality

$$\begin{aligned}\frac{1}{\mu(\mathcal{I}_r\setminus I)}\int_{\mathcal{I}_r\setminus I}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)\\ \geq \frac{1}{\mu(\mathcal{I}_r\setminus I)}\int_{\Gamma_r^\varepsilon}\left\|f_{(\omega,c)}(t)\right\|d\mu(t)\geq\varepsilon\frac{\mu(\Gamma_r^\varepsilon)}{\mu(\mathcal{I}_r\setminus I)}.\end{aligned} \tag{38}$$

Assume that assumption 2 holds, we obtain assumption 3 when making $r\longrightarrow+\infty$.

The proof is complete. □

Now, we intend to prove that $\mathscr{E}_{(\omega,c)}(\mathbb{R},\mathbb{X},\mu)$ is translation invariant.

For $\mu\in\mathcal{M}$ and $\tau\in\mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R},\mathscr{B})$ defined by

$$\mu_\tau(A)=\mu(\{a+\tau : a\in A\}),\text{for } A\in\mathscr{B}. \tag{39}$$

We need to formulate the following hypothesis for $\mu\in\mathcal{M}$ (see [4] for more details), and we also recall two important lemmas.

(H1). For all $\in\mathbb{R}$, there exists $\beta>0$ and a bounded interval I such that

$$\mu(\{a+\tau : a\in A\})\leq\beta\mu(A),\text{when } A\in\mathscr{B}\text{ satisfies } A\cap I=\varnothing. \tag{40}$$

Lemma 23. (see [4]). *Let $\mu\in\mathcal{M}_B$. Then, the measures μ and μ_τ are equivalent for all $\tau\in\mathbb{R}$.*

Lemma 24. (see [4]). *(H1) implies that for all $\alpha>0$,*

$$\limsup_{r\longrightarrow+\infty}\left(\frac{\mu([-r-\alpha,r+\alpha])}{\mu([-r,r])}\right)<+\infty. \tag{41}$$

In the following, we denote by $\mathcal{M}_{\mathrm{B}}$ the collection of measures in $\mathcal{M}$ satisfying $(H1)$.

We can prove the following result.

Theorem 25. *Let $\mu\in\mathcal{M}_B$. Then, $\mathscr{E}_{(\omega,c)}(\mathbb{R},\mathbb{X},\mu)$ is translation invariant.*

Proof. Let $f\in\mathscr{E}_{(\omega,c)}(\mathbb{R},\mathbb{X},\mu)$ and $\tau\in\mathbb{R}$. We recall that according to relation (39)

$$\mu_\tau(A)\coloneqq\mu(\{a+\tau : a\in A\}),\text{for } A\in\mathscr{B}. \tag{42}$$

We recall that according to Lemma 23 it follows that μ and μ_τ are equivalent.

It comes that

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left\|f_{(\omega,c)}(t+\tau)\right\|d\mu(t)=\frac{\mu([-r-|\tau|,r+|\tau|])}{\mu([-r,r])}\\ &\cdot\frac{1}{\mu([-r-|\tau|,r+|\tau|])}\int_{[-r,r]}\left\|f_{(\omega,c)}(t+\tau)\right\|d\mu(t)\\ &=\frac{\mu([-r-|\tau|,r+|\tau|])}{\mu([-r,r])}\\ &\cdot\frac{1}{\mu([-r-|\tau|,r+|\tau|])}\int_{[-r-|\tau|,r+|\tau|]}\left\|f_{(\omega,c)}(t)\right\|d\mu_{-\tau}(t)\\ &\leq\frac{\mu([-r-|\tau|,r+|\tau|])}{\mu([-r,r])}\\ &\cdot\frac{\beta}{\mu([-r-|\tau|,r+|\tau|])}\int_{[-r-|\tau|,r+|\tau|]}\left\|f_{(\omega,c)}(t)\right\|d\mu(t),\end{aligned} \tag{43}$$

where $\beta>0$ is a constant insuring the equivalence between μ and μ_τ.

Since $f\in\mathscr{E}_{(\omega,c)}(\mathbb{R},\mathbb{X},\mu)$, the proof is complete. □

We end this section by giving a convolution theorem for μ-(ω,c)-ergodic functions.

Theorem 26. *Let $\mu\in\mathcal{M}_B$. If $f\in\mathscr{E}_{(\omega,c)}(\mathbb{R},\mathbb{X},\mu)$ and $\widetilde{\phi}(\cdot)\coloneqq(c^\wedge(-\cdot)\phi(\cdot))\in L^1(\mathbb{R},\mathscr{L}(\mathbb{X}))$, then the convolution product of $f\mathring{a}\phi$ defined by*

$$(f\star\phi)(t)=\int_{-\infty}^{+\infty}\phi(s)f(t-s)ds,\textit{for } t\in\mathbb{R} \tag{44}$$

is μ-(ω,c)-ergodic.

Proof. Let $f\in\mathscr{E}_{(\omega,c)}(\mathbb{R},\mathbb{X},\mu)$. By Theorem 19, $(f\star\phi)\in BC(\mathbb{R},\mathbb{X},c)$. Now, we set $\mathcal{I}_r=[-r,r]$ for any $r\geq0$.

One can note that there exists $\gamma\geq0$ such that $\mu(\mathcal{I}_r)>0$ for all $r\geq\gamma$. In the other hand, one has

$$\begin{aligned}&\frac{1}{\mu(\mathcal{I}_r)}\int_{\mathcal{I}_r}\left\|(f\star\phi)_{(\omega,c)}(t)\right\|d\mu(t)\\ &\leq\frac{1}{\mu(\mathcal{I}_r)}\int_{\mathcal{I}_r}\int_{-\infty}^{+\infty}\left\|\widetilde{\phi}(s)\right\|\left\|c^\wedge(-t+s)f(t-s)\right\|dsd\mu(t),\end{aligned} \tag{45}$$

where $\widetilde{\phi}\coloneqq\phi_{(\omega,c)}\in L^1(\mathbb{R},\mathscr{L}(\mathbb{X}))$.

Applying Fubini's Theorem, it comes that

$$\begin{aligned}&\frac{1}{\mu(\mathcal{I}_r)}\int_{\mathcal{I}_r}\left\|(f\star\phi)_{(\omega,c)}(t)\right\|d\mu(t)\\ &\leq\int_{-\infty}^{+\infty}\frac{\left\|\widetilde{\phi}(s)\right\|}{\mu(\mathcal{I}_r)}\int_{\mathcal{I}_r}\left\|f_{(\omega,c)}(t-s)\right\|d\mu(t)ds.\end{aligned} \tag{46}$$

Invoking Theorem 25, we have that $\lim_{r\longrightarrow+\infty} 1/\mu(\mathcal{I}_r) \int_{\mathcal{I}_r} \|f_{(\omega,c)}(t-s)\| d\mu(t) = 0$, for all $s \in \mathbb{R}$.

Since

$$0 \le \frac{\left\|\widetilde{\phi}(s)\right\|}{\mu(\mathcal{I}_r)} \int_{\mathcal{I}_r} \left\| f_{(\omega,c)}(t-s) \right\| d\mu(t) \le \left\|\widetilde{\phi}(s)\right\| \|f\|_{(\omega,c)} \tag{47}$$

by the Lebesgue Dominated Convergence Theorem, we conclude that

$$\lim_{r\longrightarrow+\infty} \frac{1}{\mu(\mathcal{I}_r)} \int_{\mathcal{I}_r} \left\| (f\star\phi)_{(\omega,c)}(t) \right\| d\mu(t) = 0. \tag{48}$$

□

Now, we are ready to define measure (ω, c)-pseudo-almost periodic functions.

3.2. Measure (ω, c)-Pseudo-Almost Periodic Function. In this subsection, we introduce the new class of measure (ω, c)-pseudo-almost periodic function, and we study some properties of such functions. Let us define this new notion.

Definition 27. Let $\mu \in \mathcal{M}$. A function $\mathrm{f} \in C(\mathbb{R}, \mathbb{X})$ is said to be measure (ω, c)-pseudo-almost periodic (or μ-(ω, c)-pseudo-almost periodic) if f can be written in the form

$$f = g + \varphi, \tag{49}$$

where $g \in \mathrm{AP}_{\omega,c}(\mathbb{X})$ and $\varphi \in \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$.

We denote the space of all such functions by $\mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$.

We will say that g is the (ω, c)-almost periodic part of f and φ the μ-(ω, c)-pseudoergodic perturbation of f.

We have the following space inclusions:

$$\mathrm{P}_{(\omega,c)}(\mathbb{R}, \mathbb{X}) \subset \mathrm{AP}_{\omega,c}(\mathbb{X}) \subset \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu) \subset \mathrm{BC}(\mathbb{R}, \mathbb{X}, c). \tag{50}$$

Remark 28. Observe that $\mathrm{AP}_{\omega,c}(\mathbb{X})$ is a proper subspace of $\mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ since the function

$$\phi(t) = (0.2)^t \left(\sin^2(\pi t) + \sin^2\left(\sqrt{5}t\right) + e^{-t^t \cos^2(t)} \right) \in \mathrm{PAP}_{(1,0.2)}(\mathbb{R}, \mathbb{X}, \mu), \tag{51}$$

but $\phi \notin \mathrm{AP}_{(1,0.2)}(\mathbb{X})$ since $[t \mapsto \sin^2(\pi t) + \sin^2(\sqrt{5}t) + e^{-t^t \cos^2(t)}] \notin \mathrm{AP}(\mathbb{X})$.

The following theorem gives a characterization of the measure (ω, c)-pseudo-almost periodic functions.

Theorem 29. *Let $f \in C(\mathbb{R}, \mathbb{X})$. Then, $f \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ if and only if*

$$f(t) \equiv c^\wedge(t) u(t), \textit{ with } c^\wedge(t) \coloneqq c^{t/\omega} \textit{ and } u \in PAP(\mathbb{R}, \mathbb{X}, \mu). \tag{52}$$

Proof. Obviously, if $f(t) = c^\wedge(t)u(t)$ with $u \in \mathrm{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ then $f \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$.

Conversely, let $f \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$. Then, $\exists (g, \varphi) \in \mathrm{AP}_{\omega,c}(\mathbb{X}) \times \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ such that $f = g + \varphi$. Therefore, taking $u(t) \coloneqq c^\wedge(-t) f(t)$, it comes that $u \in \mathrm{PAP}(\mathbb{R}, \mathbb{X}, \mu)$. □

In view of Definition 27, for any $f \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$, we say that $c^\wedge(t)u(t)$ is the c-factorization of f.

We give the first basic result.

Proposition 30. *Let $\mu \in \mathcal{M}$. Then, $PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a vector space.*

Proof. Obvious. □

Now, we intend to show that $\mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu) = \mathrm{AP}_{\omega,c}(\mathbb{X}) \oplus \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$. In order to prove Proposition 30, we will need following lemma.

Lemma 31. *Assume $f \in AP_{\omega,c}(\mathbb{X})$, write*

$$B_\varepsilon \coloneqq \left\{ \tau \in \mathbb{R} : \left\| f_{(\omega,c)}(t_0 + \tau) - f_{(\omega,c)}(t_0) \right\| < \varepsilon \right\}, \tag{53}$$

where $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ is fixed. Then, there exists $s_1, s_2, \cdots, s_m \in \mathbb{R}$ such that

$$\bigcup_{i=1}^{m} (s_i + B_\varepsilon) = \mathbb{R}. \tag{54}$$

We have following result.

Proposition 32. *Let $\mu \in \mathcal{M}_B$ and $f \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ be such that*

$$f = g + \varphi, \tag{55}$$

where g is its (ω, c)-almost periodic component, and then we have

$$g(\mathbb{R}) \subset f(\bar{\mathbb{R}}). \tag{56}$$

Therefore, $\|f\|_{(\omega,c)} \ge \|g\|_{(\omega,c)} \ge \inf_{t\in\mathbb{R}} |g_{(\omega,c)}(t)| \ge \inf_{t\in\mathbb{R}} |f_{(\omega,c)}(t)|$.

Proof. Suppose that (56) is not true, then there exists $t_0 \in \mathbb{R}$, $\varepsilon > 0$ such that

$$\left\| g_{(\omega,c)}(t_0) - f_{(\omega,c)}(t) \right\| \ge 2\varepsilon, t \in \mathbb{R}. \tag{57}$$

Let $s_1, s_2, \cdots, s_m$ be as in Lemma 31 and write

$$\tau_i = s_i - t_0, i = 1, 2, \cdots, m, \eta = \max_{1\le i\le m} |\tau_i|. \tag{58}$$

For $r \in \mathbb{R}$ with $|r| > \eta$, we let

$$B_{\varepsilon,r}^{(i)} \coloneqq [-r + \eta - \tau_i, r - \eta - \tau_i] \cap (t_0 + B_\varepsilon), i = 1, 2, \cdots, m, \tag{59}$$

where B_ε is as in Lemma 31. It is clear that

$$\bigcup_{i=1}^{m} \left(\tau_i + B_{\varepsilon,r}^{(i)}\right) = [-r + \eta, r - \eta]. \tag{60}$$

Thus, we obtain

$$\begin{aligned} 2(r-\eta) = \mu([-r+\eta, r-\eta]) &\le \sum_{i=1}^{m} \mu\left(\tau_i + B_{\varepsilon,r}^{(i)}\right) \\ &= \sum_{i=1}^{m} \mu\left(B_{\varepsilon,r}^{(i)}\right) \le m \cdot \max_{1\le i\le m} \left\{\mu\left(B_{\varepsilon,r}^{(i)}\right)\right\} \\ &\le m \cdot \mu([-r, r] \cap (t_0 + B_\varepsilon)), \end{aligned} \tag{61}$$

since for each $i = 1, 2, \cdots, m$,

$$B_{\varepsilon,r}^{(i)} \subset ([-r, r] \cap (t_0 + B_\varepsilon)). \tag{62}$$

Using inequality (57), we have

$$\begin{aligned} \left\|\phi_{(\omega,c)}(t)\right\| = \left\|f_{(\omega,c)}(t) - g_{(\omega,c)}(t)\right\| &\ge \left\|g_{(\omega,c)}(t_0) - f_{(\omega,c)}(t)\right\| \\ &- \left\|g_{(\omega,c)}(t) - g_{(\omega,c)}(t_0)\right\| > \varepsilon, \end{aligned} \tag{63}$$

any $t \in t_0 + B_\varepsilon$.
This and inequality (61) together give

$$\frac{1}{\mu([-r, r])} \int_{[-r,r]} \left\|\phi_{(\omega,c)}(t)\right\| d\mu(t) \ge \frac{r-\eta}{mr} \varepsilon \longrightarrow \frac{\varepsilon}{m}, \text{as } r \longrightarrow \infty. \tag{64}$$

This is a contradiction since $h \in \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ and establishes our claim (56). □

We can now establish the uniqueness of the decomposition in Definition 27.

Theorem 33. *Let $\mu \in \mathscr{M}_B$. Then, the decomposition (49) is unique.*

Proof. Let $f \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$.

Assume that f admits both decomposition $f = g_1 + \varphi_1$ and $f = g_2 + \varphi_2$, then $0 = (g_1 - g_2) + (\varphi_1 - \varphi_2)$.

Since $g_1 - g_2 \in \mathrm{AP}_{\omega,c}(\mathbb{X})$ and $\varphi_1 - \varphi_2 \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$, in view of Proposition 32, we deduce that $g_1 - g_2 = 0$ and consequently, $\varphi_1 = \varphi_2$ which proves the uniqueness of the decomposition. □

From above, it is clear that

$$\mathrm{AP}_{\omega,c}(\mathbb{X}) \cap \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu) = \{0\}. \tag{65}$$

Furthermore, we have following results.

Theorem 34. *The space $PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a translation invariant.*

Proof. This is a direct consequence of Proposition 9, Theorem 25, and Theorem 33. □

Theorem 35. *The space $PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a translation invariant C^*-subalgebra of $BC(\mathbb{R}, \mathbb{X}, c)$. Furthermore,*

$$\frac{PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)}{\mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)} \cong AP_{\omega,c}(\mathbb{X}). \tag{66}$$

Proof. We show that $\mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ is a closed subspace of $\mathrm{BC}(\mathbb{R}, \mathbb{X}, c)$.

Let $(f_n) \subset \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ be Cauchy. By proposition 32, the sequence $(g_n) \subset \mathrm{AP}_{\omega,c}(\mathbb{X})$ is Cauchy too and so is $(\varphi_n) \subset \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$. Since $\mathrm{AP}_{\omega,c}(\mathbb{X})$ and $\mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ are closed in $\mathrm{BC}(\mathbb{R}, \mathbb{X}, c)$, there are $g \in \mathrm{AP}_{\omega,c}(\mathbb{X})$ and $\varphi \in \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ such that $\|(g_n)_{(\omega,c)} - g_{(\omega,c)}\| \longrightarrow 0$ and $\|(\varphi_n)_{(\omega,c)} - \varphi_{(\omega,c)}\| \longrightarrow 0$ as $n \longrightarrow \infty$. Set $f = g + \varphi$, then $f \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ and $\|(f_n)_{(\omega,c)} - f_{(\omega,c)}\| \longrightarrow 0$ as $n \longrightarrow 0$.

The rest of the proof is clear. □

Now, we show the completeness of $\mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ with the following result.

Theorem 36. *Let $\mu \in \mathscr{M}_B$. Then, $(PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_{(\omega,c)})$ is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $\mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$. Then, given $\varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that for all $p, q \ge n_0$,

$$\left\|\left(f_p\right)(t) - \left(f_q\right)(t)\right\|_{(\omega,c)} < \varepsilon. \tag{67}$$

Invoking Theorem 29, $\exists (u_p, u_q) \in (\mathrm{PAP}(\mathbb{R}, \mathbb{X}, \mu))^2$ such that $f_p(t) = c^\wedge(t) u_p(t)$ and $f_q(t) = c^\wedge(t) u_q(t)$ for all $t \in \mathbb{R}$ and since $p, q \ge n_0$, we have

$$\|u_p(t) - u_q(t)\| \le \left\|f_p(t) - f_q(t)\right\|_{(\omega,c)} < \varepsilon. \tag{68}$$

Consequently, let (u_n) be a Cauchy sequence in $\mathrm{PAP}(\mathbb{R}, \mathbb{X}, \mu)$. Using the completeness of $\mathrm{PAP}(\mathbb{R}, \mathbb{X}, \mu)$, we know that $\exists u \in \mathrm{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ such that $\|u_n - u\| \longrightarrow 0$ as $n \longrightarrow \infty$.

We take $f(t) := c^{\wedge}(t)u(t)$. We claim that $\|u_n - u\| \longrightarrow 0$ as $n \longrightarrow \infty$. And it can be easily seen that

$$\|f_n - f\|_{(\omega,c)} = \sup_{t\in\mathbb{R}}\|u_n - u\| \longrightarrow 0\ (n\longrightarrow\infty), \quad (69)$$

which completes the proof. □

We end this subsection giving a general convolution theorem for our new class of functions.

Theorem 37. *Let $\mu \in \mathscr{M}_B$. If $f \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ and $\widetilde{\phi} := (c^{\wedge}(-\cdot)\phi(\cdot)) \in L^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$, then the convolution product of $f\star\phi$ defined by*

$$(f\star\phi)(t) = \int_{-\infty}^{+\infty} \phi(s)f(t-s)ds, for\ t \in \mathbb{R} \quad (70)$$

is μ-(ω, c)-pseudo-almost periodic.

Proof. Let $f \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$ and $\widetilde{\phi} := (c^{\wedge}(-\cdot)\phi(\cdot)) \in \mathrm{L}^1(\mathbb{R}, \mathscr{L}(\mathbb{X}))$.

First, note that using Theorem 19, $(f\star\phi) \in \mathrm{BC}(\mathbb{R}, \mathbb{X}, c)$. Furthermore, according to Theorem 29, there exists a $u \in \mathrm{PAP}(\mathbb{R}, \mathbb{X}, \mu)$ such that $f(t) := c^{\wedge}(t)u(t)$, for any $t \in \mathbb{R}$. It comes that

$$\begin{aligned}(f\star\phi)(t) &= \int_{-\infty}^{+\infty} \phi(s)f(t-s)ds = c^{\wedge}(t)\int_{-\infty}^{+\infty} c^{\wedge}(-s)\phi(s)c^{\wedge}(-t+s)f(t-s)ds \\ &= c^{\wedge}(t)\int_{-\infty}^{+\infty} \widetilde{\phi}(s)u(t-s)ds.\end{aligned} \quad (71)$$

Invoking successively [[4], Theorem 22, pp. 511], and our Theorem 29, we have that (71) is μ-(ω, c)-pseudo-almost periodic. The proof is complete. □

Example 38. The unique solution of the heat equation

$$u_t(x, t) = u_{xx}(x, t), x \in \mathbb{R}, t \geq 0, \quad (72)$$

with the initial condition $u(x, 0) = f(x)$ is given by

$$u(x, t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} e^{-(x-s)^2/4t} f(s)ds, x \in \mathbb{R}, t \geq 0. \quad (73)$$

If $c^{\wedge}(-\cdot)e^{-(\cdot)^2/4t_0} \in \mathrm{L}^1(\mathbb{R})$ and $f \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}, \mu)$, then by Theorem 26, the solution

$$[x \mapsto u(x, t_0), x \in \mathbb{R}] \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}, \mu). \quad (74)$$

4. Jointly Continuous Case

This section is devoted to the study of a composition result well suited for the introduced (ω, c)-periodicity concept. The main results of this section are Theorems 45 and 51. But first, let us define some new notions.

First of all, reader should be aware that the already known concept of compactness for subsets seems to be irrelevant when it comes to deal with (ω, c)-periodicity where $c \neq 1$ since (ω, c) periodic type functions are not bounded on $\mathbb{R}$ (i.e., 1-bounded on $\mathbb{R}$) but c-bounded on $\mathbb{R}$.

With the following definition, we propose a new concept of compactness for subset well suited for (ω, c)-periodic calculus.

Definition 39. Let $\mathscr{K}$ be a nonempty set. We say that $\mathscr{K}$ is an (ω, c)-type compact subset of $\mathbb{Y}$ if and only if following assumptions are satisfied:

(1) $\mathscr{K}$ is compact

(2) Every $k \in \mathscr{K}$ admits following decomposition $k := c^{\wedge}(-\cdot)y \in \mathscr{K}$ where $y \in \mathbb{Y}$

One can note that a compact subset of $\mathrm{AP}(\mathbb{X})$ is in fact an (ω, c)-type subset of $\mathrm{AP}_{\omega,c}(\mathbb{X})$ since if $\mathscr{K}$ is a compact subset of $\mathrm{AP}(\mathbb{X})$, we have the following equality:

$$\begin{aligned}\Sigma &:= \{u(t)\colon t \in \mathbb{R}, u \in \mathscr{K}\} \\ &= \left\{\left(c^{\wedge}(t)u(t)\right)_{(\omega,c)} : t \in \mathbb{R}, \left(c^{\wedge}(t)u(t)\right) \in \mathrm{AP}_{\omega,c}(\mathbb{X}), u \in \mathscr{K}\right\},\end{aligned} \quad (75)$$

for any $c \in \mathbb{C} - \{0\}$.

4.1. On (ω, c)-Almost Periodic Functions Depending on a Parameter. Throughout this section, we introduce a new concept of (ω, c)-almost periodic function in the jointly continuous case. Then, we study some properties and establish some results as the continuity of Nemytskii's superposition operator.

Definition 40 (see [20]). A function $\mathrm{F} \in \mathrm{C}(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called (Yoshizawa) almost periodic in $\mathrm{t} \in \mathbb{R}$ uniformly in $\mathrm{y} \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any compact $\mathrm{K} \subset \mathbb{Y}$, and there exists $\mathrm{L} > 0$, such that for all $\beta \in \mathbb{R}$, there exists $\tau \in [\beta, \beta + \mathrm{L}]$ with

$$\sup_{t\in\mathbb{R}}\sup_{y\in K}\|F(t+\tau, y) - F(t, y)\| < \varepsilon, \quad (76)$$

for all $t \in \mathbb{R}$ and all $y \in K$.

The collection of such functions will be denoted by AP $(\mathbb{Y}, \mathbb{X})$.

Such number τ in (76) is called ε-translation number of $F(t, y)$, and we denote by $\mathscr{T}_\varepsilon(F, K)$ the set of all ε-translation numbers of F for $y \in K$. This set has the following properties:

For a fixed compact set K,

(1) An ε-translation number is also an ε'-translation number if $\varepsilon' > \varepsilon$, and hence we have the inclusion $\mathscr{T}_\varepsilon(F, K) \subset \mathscr{T}_{\varepsilon'}(F, K)$

(2) If $\tau \in \mathscr{T}_\varepsilon(F, K)$, then $-\tau \in \mathscr{T}_\varepsilon(F, K)$

(3) If $(\tau_1, \tau_2) \in \mathscr{T}_{\varepsilon_1}(F, K) \times \mathscr{T}_{\varepsilon_2}(F, K)$, then $\tau_1 + \tau_2 \in \mathscr{T}_{\varepsilon_1+\varepsilon_2}(F, K)$

In what follows, we assume that $\omega \in \mathbb{R}_+^* \cap \mathscr{T}_\varepsilon(F, K)$.

In [17], authors have introduced two concepts of (ω, c)-almost periodic functions in the case of jointly continuous functions, but in this paper, it uses a novel approach.

Definition 41. A function $\mathrm{F} \in \mathrm{C}(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called (ω, c)-almost periodic in $\mathrm{t} \in \mathbb{R}$ uniformly in $\mathrm{y} \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any (ω, c)-type compact subset $\mathscr{K}$ of $\mathbb{Y}$, there exists $\mathrm{L} > 0$, such that for all $\beta \in \mathbb{R}$, there exists $\tau \in [\beta, \beta + \mathrm{L}]$ with

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \sup_{y\in\mathscr{K}} \left\| c\hat{}(-t-\tau)F(t+\tau, y) - c\hat{}(-t)F(t, y) \right\| < \varepsilon, \\ &\Leftrightarrow \sup_{t\in\mathbb{R}} \sup_{y\in\mathscr{K}} \left\| F_{(\omega,c)}(t+\tau, y) - F_{(\omega,c)}(t, y) \right\| < \varepsilon, \end{aligned} \tag{77}$$

for all $t \in \mathbb{R}$ and all $y \in K$, where $F_{(\omega,c)}(t, \cdot) := c^{-t/\omega} F(t, \cdot)$.

The space of all such functions will be denoted by $\mathrm{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$.

In the following, we use the notation: $F_{(\omega,c)}(t, \cdot) := c^{-t/\omega} F(t, \cdot)$.

Remark 42. When $c = 1$, $\mathrm{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X}) = \mathrm{AP}(\mathbb{Y}, \mathbb{X})$.

Proposition 43. *$(AP_{\omega,c}(\mathbb{Y}, \mathbb{X}), \|\cdot\|_{(\omega,c)})$ is a Banach space.*

We need to develop some tools in order to propose a composition theorem for measure (ω, c)-pseudo-almost periodic functions.

We give the following results.

Lemma 44. *If $\mathscr{K}$ is an (ω, c)-type compact subset of $AP_{\omega,c}(\mathbb{Y})$, then*

$$\Sigma := \left\{ u_{(\omega,c)}(t) \colon t \in \mathbb{R}, u \in AP_{\omega,c}(\mathbb{Y}), u_{(\omega,c)} \in \mathscr{K} \right\} \tag{78}$$

is a relatively compact subset of $\mathbb{Y}$.

Proof. Let $\varepsilon > 0$.

Since $\mathscr{K}$ is compact, it is also precompact; thus, there exists $\{(u_1)_{(\omega,c)}, \cdots, (u_m)_{(\omega,c)}\}$ a finite (ω, c)-type subset of $\mathrm{AP}_{\omega,c}(\mathbb{Y})$ (i.e., a finite subset of $\mathrm{AP}(\mathbb{Y})$) such that

$$\mathscr{K} \subset \bigcup_{1\le i\le m} \left\{ z \in \mathrm{AP}_{\omega,c}(\mathbb{Y}) \colon \|z - u_i\|_{(\omega,c)} \le \frac{\varepsilon}{2} \right\}. \tag{79}$$

Since, $(u_i)_{(\omega,c)}(\mathbb{R})$ is relatively compact in $\mathbb{X}$ for all $i = 1, \cdots, m$, then $\bigcup_{1\le i\le m}(u_i)_{(\omega,c)}(\mathbb{R})$ is also relatively compact and consequently, there exists a finite subset of $\mathbb{R}$ $\{t_1, \cdots, t_k\}$ such that

$$\bigcup_{1\le i\le m} (u_i)_{(\omega,c)}(\mathbb{R}) \subset \bigcup_{1\le i\le m} \bigcup_{1\le j\le k} \left\{ y \in \mathbb{Y} : \left\| y - (u_i)_{(\omega,c)}(t_j) \right\| \le \frac{\varepsilon}{2} \right\}. \tag{80}$$

If $y \in \mathbb{Y}$, there exists $z \in \mathscr{K}$ and $t \in \mathbb{R}$ such that $y = z(t)$, and there exists $i \in \{1, \cdots, m\}$ such that $\|z - (u_i)_{(\omega,c)}\| \le \varepsilon/2$ and consequently $\|y - (u_i)_{(\omega,c)}(t)\| \le \varepsilon/2$.

Now, using the previous inclusion, there exists $p \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, k\}$ such that

$$\left\| (u_i)_{(\omega,c)}(t) - (u_p)_{(\omega,c)}(t_j) \right\| \le \frac{\varepsilon}{2}. \tag{81}$$

It comes that

$$\begin{aligned} \left\| y - (u_i)_{(\omega,c)}(t_j) \right\| &\le \left\| y - (u_i)_{(\omega,c)}(t) \right\| \\ &+ \left\| (u_i)_{(\omega,c)}(t) - (u_p)_{(\omega,c)}(t_j) \right\| \le \frac{2\varepsilon}{2} = \varepsilon. \end{aligned} \tag{82}$$

□

This proves that $\Sigma \subset \bigcup_{1\le i\le m} \bigcup_{1\le j\le k} \{y \in \mathbb{Y} : \|y - (u_i)_{(\omega,c)}(t_j)\| \le \varepsilon/2\}$ or in other words, Σ is precompact, and since $\mathbb{Y}$ is complete, we obtain that Σ is relatively compact.

Now, for a given function $F \in \mathrm{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$, we define Nemytskii's superposition operator $\mathscr{N}_F : \mathrm{AP}_{\omega,c}(\mathbb{Y}) \longrightarrow \mathrm{AP}_{\omega,c}(\mathbb{X})$ such that $[t \mapsto u(t)] \mapsto \mathscr{N}_F(u) := [t \mapsto F(t, u(t))]$.

The first main result of this section is the following theorem.

Theorem 45. *Let $F \in AP_{\omega,c}(\mathbb{Y}, \mathbb{X})$. Then, the Nemytskii superposition operator $\mathscr{N}_F$ is continuous from $AP_{\omega,c}(\mathbb{Y})$ into $AP_{\omega,c}(\mathbb{X})$.*

Proof. Let $\mathscr{K}$ be an (ω, c)-type compact subset of $\mathrm{AP}_{\omega,c}(\mathbb{Y})$ let $\tilde{u} \in \mathscr{K}$ and $\varepsilon > 0$.

We set $\Sigma := \{u_{(\omega,c)}(t) \colon t \in \mathbb{R}, u \in \mathrm{AP}_{\omega,c}(\mathbb{Y}), u_{(\omega,c)} \in \mathscr{K}\}$. According to Lemma 44, the closure $\bar{\Sigma}$ is compact.

Since $F \in \mathrm{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$, there exists $l > 0$ such that for $\alpha \in \mathbb{R}$, there exists $-\tau \in [\alpha, \alpha + l]$ satisfying

$$\left\| F_{(\omega,c)}(t+\tau, y) - F_{(\omega,c)}(t, y) \right\| \le \frac{\varepsilon}{3}, \text{for } all\ (y, t) \in \Sigma \times \mathbb{R}. \tag{83}$$

Since $\bar{\Sigma} \times [0, l]$ is compact, then $F_{(\omega,c)}$ is uniformly continuous on it and consequently, there exists $\delta > 0$ such that, for all $t_1, t_2 \in [0, l]$ and for all $y_1, y_2 \in \bar{\Sigma}$

$$(\|y_1 - y_2\| \le \delta, |t_1 - t_2| \le \delta) \Rightarrow \left\| F_{(\omega,c)}(t_1, y_1) - F_{(\omega,c)}(t_2, y_2) \right\| \le \frac{\varepsilon}{3}. \tag{84}$$

And this implies that

$$\|y_1 - y_2\| \le \delta \Rightarrow \left\| F_{(\omega,c)}(t, y_1) - F_{(\omega,c)}(t, y_2) \right\| \le \frac{\varepsilon}{3}, \text{ for all } t \in [0, l], \tag{85}$$

if $\tilde{v} \in \mathscr{K}$ satisfies $\|\tilde{v} - \tilde{u}\|_\infty \le \delta \Leftrightarrow \|c^\wedge(t)\tilde{v}(t) - c^\wedge(t)\tilde{u}(t)\|_{(\omega,c)} \le \delta$.

We set $u(t) \coloneqq c^\wedge(t)\tilde{u}(t)$ and $v(t) \coloneqq c^\wedge(t)\tilde{v}(t)$ for all $t \in \mathbb{R}$.

Then, we have $u, v \in \mathrm{AP}_{\omega,c}(\mathbb{Y})$, and using (83) and (85), we obtain, for all $t \in \mathbb{R}$

$$\begin{aligned} &\left\| F_{(\omega,c)}(t, u(t)) - F_{(\omega,c)}(t, v(t)) \right\| \\ &\quad \le \left\| F_{(\omega,c)}(t, u(t)) - F_{(\omega,c)}(t - \tau, u(t)) \right\| \\ &\quad + \left\| F_{(\omega,c)}(t - \tau, u(t)) - F_{(\omega,c)}(t - \tau, v(t)) \right\| \\ &\quad + \left\| F_{(\omega,c)}(t - \tau, v(t)) - F_{(\omega,c)}(t, v(t)) \right\| \le 3\frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{86}$$

And so, by taking the supremum on the $t \in \mathbb{R}$, we obtain $\|\mathscr{N}_F(u) - \mathscr{N}_F(v)\|_{(\omega,c)} \le \varepsilon$.

This proves that the restriction of $\mathscr{J}_f$ to $\mathscr{K}$ is continuous for all (ω, c)-type compact subset $\mathscr{K}$ of $\mathrm{AP}_{\omega,c}(\mathbb{Y})$. And since $(\mathrm{AP}_{\omega,c}(\mathbb{Y}), \|\cdot\|_{(\omega,c)})$ and $(\mathrm{AP}_{\omega,c}(\mathbb{X}), \|\cdot\|_{(\omega,c)})$ are Banach Spaces, this proves the continuity of $\mathscr{J}_F$ on $\mathrm{AP}_{\omega,c}(\mathbb{Y})$. □

The following proposition is a generalization of Cieutat, Fatajou, and N'Gu´er´ekata´s Theorem in [21] which becomes the particular case $c = 1$ of our result.

Proposition 46. *Let $F : \mathbb{R} \times \mathbb{Y} \longrightarrow \mathbb{X}$ be a continuous function. Then, $F \in AP_{\omega,c}(\mathbb{Y}, \mathbb{X})$ if and only if the following conditions hold:*

(1) For all $y \in \mathbb{Y}$, $F_{(\omega,c)}(\cdot, y) \in AP(\mathbb{X})$

(2) $F_{(\omega,c)}$ is uniformly continuous on each (ω, c)-type compact set $\mathscr{K}$ in $\mathbb{Y}$ with respect to the second variable, namely, for each (ω, c)-type compact set $\mathscr{K}$ in $\mathbb{Y}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all y_1, $y_2 \in \mathscr{K}$, one has

$$\|y_1 - y_2\| \le \delta \Rightarrow \sup_{t\in\mathbb{R}} \left\| F_{(\omega,c)}(t, y_1) - F_{(\omega,c)}(t, y_2) \right\| \le \varepsilon. \tag{87}$$

4.2. Measure (ω, c)-Pseudo-Almost Periodic Functions Depending on a Parameter and Composition Principle. In this section, we extend our new concept of measure (ω, c)-pseudo-almost periodic functions to that of measure (ω, c)-pseudo-almost periodic functions depending on a parameter.

Here, we propose a concept of μ-(ω, c)-ergodicity for the jointly continuous functions case.

Definition 47. Let $\mu \in \mathscr{M}$. A function $\mathrm{F} \in \mathrm{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mathrm{c})$ is said to be μ-(ω, c)-ergodic in t uniformly with respect to y $\in \mathbb{Y}$ if the two following conditions are true:

(1) For all $y \in \mathbb{Y}$, $F_{(\omega,c)}(\cdot, y) \in \mathscr{E}(\mathbb{R}, \mathbb{X}, \mu)$

(2) $F_{(\omega,c)}$ is uniformly continuous on each (ω, c)-type compact set $\mathscr{K}$ in $\mathbb{Y}$ with respect to the second variable, namely, for each (ω, c)-type compact set $\mathscr{K}$ in $\mathbb{Y}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y_1, y_2 \in \mathscr{K}$, one has

$$\|y_1 - y_2\| \le \delta \Rightarrow \sup_{t\in\mathbb{R}} \left\| F_{(\omega,c)}(t, y_1) - F_{(\omega,c)}(t, y_2) \right\| \le \varepsilon. \tag{88}$$

We denote the space of all such functions by $\mathscr{E}_{\omega,c,3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Remark 48. When $\mathrm{c} = 1$, we write $\mathscr{E}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ instead of $\mathscr{E}_{\omega,1,3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Now, we are able to introduce the new concept of measure (ω, c)-pseudo-almost periodic functions depending on a parameter.

Definition 49. Let $\mu \in \mathscr{M}$. A function $\mathrm{F} \in \mathrm{C}(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is said to be μ-(ω, c)-pseudo-almost periodic in t uniformly with respect to y $\in \mathbb{Y}$ if F is written in the form

$$F = G + \Phi, \tag{89}$$

where $G \in \mathrm{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X})$ and $\Phi \in \mathscr{E}_{\omega,c,3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

$\mathrm{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ denotes the set of such that functions.

The following inclusion hold

$$\mathrm{AP}_{\omega,c}(\mathbb{Y}, \mathbb{X}) \subset \mathrm{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu) \subset \mathrm{BC}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, c). \tag{90}$$

Remark 50. When $\mathrm{c} = 1$, we write $\mathrm{PAP}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ instead of $\mathrm{PAP}_{\omega,1}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

As in the previous section, we propose a characterization result which holds for (ω, c)-almost periodic, μ-(ω, c)-ergodic and μ-(ω, c)-pseudo-almost periodic functions in t uniformly with respect to $y \in \mathbb{Y}$.

Theorem 51. *Let $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.*

Then, $F \in PAP_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ (resp., $AP_{\omega,c}(\mathbb{Y}, \mathbb{X})$ or $\mathscr{E}_{\omega,c,3}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$) if and only if

$$F(t, y) \equiv c^\wedge(t)u(t, y), \tag{91}$$

with $c^\wedge(t) \coloneqq c^{t/\omega}$ and $u \in PAP(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ (resp., $AP(\mathbb{Y}, \mathbb{X})$ or $\mathscr{E}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$).

Proof. The proof is similar to the one of Theorem 29. □

Using Proposition 46 and Definition 47, one can obtain following.

Theorem 52. *Let $\mu \in \mathscr{M}$ and $F : \mathbb{R} \times \mathbb{Y} \longrightarrow$ be μ-(ω, c)-ergodic in t uniformly with respect to $y \in \mathbb{Y}$. Then,*

(1) For all $y \in \mathbb{Y}$, $F_{(\omega,c)}(\cdot, y) \in PAP(\mathbb{R}, \mathbb{X}, \mu)$

(2) $F_{(\omega,c)}$ is uniformly continuous on each (ω, c)-type compact set $\mathscr{K}$ in $\mathbb{Y}$ with respect to the second variable

We are now in a position to give a composition theorem for measure (ω, c)-pseudo-almost periodic functions. We have the following theorem.

Theorem 53. *Let $\mu \in \mathscr{M}$, $F \in PAP_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ and $y \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{Y}, \mu)$. Assume that the following hypothesis holds.*

For all bounded subset Ω of $\mathbb{Y}$, F is c-bounded on $\mathbb{R} \times \Omega$ (i.e., $F_{(\omega,c)}$ is bounded on $\mathbb{R} \times \Omega$).

Then, $[t \mapsto F(t, y_{(\omega,c)}(t))] \in PAP_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Proof. First note that the function $[t \mapsto F(t, y_{(\omega,c)}(t))]$ is continuous and by Hypothesis (10), it is c-bounded. Since $F \in \mathrm{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$ by Theorem 5, there exists

$$\begin{aligned} &\tilde{F} \in \mathrm{PAP}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu) \text{ such that } F(t, y(t)) \\ &\quad \coloneqq c^\wedge(t)\tilde{F}(t, y(t)), \forall t \in \mathbb{R}. \end{aligned} \tag{92}$$

Now since $y_{(\omega,c)} \in \mathrm{PAP}(\mathbb{R}, \mathbb{Y}, \mu)$, and using Theorem 4.10 in [4], we deduce that the function

$$\left[t \mapsto \tilde{F}\left(t, y_{(\omega,c)}(t)\right)\right] \in \mathrm{PAP}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu). \tag{93}$$

In conclusion, invoking again Theorem 51, we showed that

$$\left[t \mapsto F\left(t, y_{(\omega,c)}(t)\right)\right] \in \mathrm{PAP}_{\omega,c}(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu). \tag{94}$$

The proof is complete. □

The following theorem will be very useful in the sequel.

Corollary 54. *Let $\mu \in \mathscr{M}$, $\Psi \in C(\mathbb{X}, \mathbb{Y})$, $\omega > 0$, and $c \in \mathbb{C} - \{0\}$.*

Assume that for all bounded subset $\mathbf{B}$ of $\mathbb{X}$, Ψ is c-bounded on $\mathbf{B}$, then if $x \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu)$,

$$\left[t \mapsto \Psi\left(x_{(\omega,c)}(t)\right) \coloneqq c^\wedge(t)\tilde{\Psi}\left(x_{(\omega,c)}(t)\right)\right] \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{X}, \mu). \tag{95}$$

Proof. This is direct consequence of Theorem 5 with $F(t, x_{(\omega,c)}(t)) = \Psi(x_{(\omega,c)}(t))$. □

5. Application: Measure (ω, c)-Pseudo-Almost Periodic Solutions to a Lasota-Wazewska Model

First, Wazewska-Czyzewska and Lasota [22] proposed in 1976 the delay logistic equations with one constant concentrated delay

$$N'(t) = -\mu N(t) + p e^{-rN(t-\tau)}, \tag{96}$$

in order to describe the survival of red blood cells in an animal. Here, $N(t)$ denotes the number of red blood cells at time t, μ is the probability of death of a red blood cell, p and r are positive constants related to the production of red blood cells per unit time, and τ is the time required to produce a red blood cell. Few years later, Gopalsamy and Trofimchuk [23] obtained that the Lasota-Wazewska model with one discrete delay

$$x'(t) = -\alpha(t)x(t) + \beta(t)e^{-\nu x(t-\tau)} \tag{97}$$

has a globally attractive almost periodic solution under some additional assumptions.

Recently, Cherif and Miraoui [24] investigate the existence, the uniqueness, the global attractivity, and the exponential stability of the measure pseudo-almost periodic solutions for the following Lasota-Wazewska model with measure pseudo-almost periodic coefficients and mixed delays

$$\begin{aligned} y'(t) = &-\alpha(t)y(t) + \sum_{j=1}^{m} a_j(t) e^{-\omega_j(t)\int_{-\infty}^{t} K_j(t-s)y(s)ds} \\ &+ \sum_{i=1}^{n} b_i(t) e^{-\beta_i(t)\sum_{j=1}^{p} y\left(t-\tau_{ij}\right)}, t \in \mathbb{R}. \end{aligned} \tag{98}$$

The aim here is to study the existence and uniqueness of a generalized Lasota-Wazewska model with μ-(ω, c)-pseudo-almost periodic coefficients and with mixed delay which is in the form:

$$\begin{aligned} y'(t) = &-\alpha(t)y(t) + \sum_{j=1}^{m} a_j(t) e^{-c^\wedge(-t)\omega_j(t)\int_{-\infty}^{t} K_j(t-s)y(s)ds} \\ &+ \sum_{i=1}^{n} b_i(t) e^{-c^\wedge(-t)\beta_i(t)\sum_{j=1}^{p} y\left(t-\tau_{ij}\right)}, t \in \mathbb{R}, \end{aligned} \tag{99}$$

where $y(t)$ stands for the number of red blood cells at time t, and $\alpha(t)$ is the average part of red blood cells population being destroyed in time t. For all $1 \leq j \leq m$ and $1 \leq i \leq n$, $a_j(t)$ and $b_i(t)$ are the connected with demand for oxygen at time t, $\omega_j(t)$ and $\beta_i(t)$ characterize excitability of haematopoietic system at time t, K_j is the probability kernel of the distributed delays, and τ_{ij} is the time required to produce a red blood cell. One can note that we consider in our new approach the μ-(ω, c)-pseudo-almost periodic for the connected with demand for oxygen at time t and the μ-pseudo-almost periodic for the excitability of haematopoietic system at time t since it is more realistic for the description of the physical and biological phenomena.

The method consists to reduce the existence of the unique solution for the Lasota-Wazewska model (99) to the search for the existence of the unique fixed point of an appropriate operator on the Banach space $\mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$.

Notice that we restrict ourselves to $\mathbb{R}_+$-valued functions since only nonnegative solutions are biologically meaningful.

5.1. Existence and Uniqueness of μ-(ω, c)-Pseudo-Almost Periodic Solution to the Model. In what follows, given a c-bounded continuous function f defined on $\mathbb{R}$, $\overline{f}_{(\omega,c)}$ and $\underline{f_{(\omega,c)}}$ are defined by

$$\overline{f}_{(\omega,c)} = \sup_{t\in\mathbb{R}} f_{(\omega,c)}(t) = \sup_{t\in\mathbb{R}} c^\wedge(-t) f(t), \text{ and } \underline{f_{(\omega,c)}} = \inf_{t\in\mathbb{R}} f_{(\omega,c)}(t) = \inf_{t\in\mathbb{R}} c^\wedge(-t) f(t). \tag{100}$$

Remark 55. If c = 1, we use the notations

$$\bar{f} \coloneqq \overline{f}_{(\omega,1)} = \sup_{t\in\mathbb{R}} f(t) \text{ and } \underline{f} \coloneqq \underline{f_{(\omega,1)}} = \inf_{t\in\mathbb{R}} f(t). \tag{101}$$

First, we give sufficient conditions which ensures existence and uniqueness of μ-(ω, c)-pseudo-almost periodic solution of (99).

(C1)$0 < c \le 1$

(C2)$\alpha \in \mathrm{AP}(\mathbb{R}_+)$

(C3)$\beta_i, \omega_j \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$, for all $(i, j) \in [[1, n]] \times [[1, m]]$

(C4)$a_j, b_i \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$, for all $(i, j) \in [[1, n]] \times [[1, m]]$

(C5)$\sum_{j=1}^m \overline{(a_j)}_{(\omega,c)} \omega_j + p\xi \sum_{i=1}^n \overline{(b_i)}_{(\omega,c)} \beta_i / \underline{\alpha} < 1$

(C6) For all $1 \le j \le m$, $(K_j)_{(\omega,c)} : [0,+\infty) \longrightarrow \mathbb{R}_+$ are continuous, integrable, and

$$\int_0^\infty (K_j)_{(\omega,c)}(u)du = 1, \text{ and } \int_0^\infty (K_j)_{(\omega,c)}(u) e^{\lambda u} du < +\infty, \tag{102}$$

where λ is a sufficiently non negative small constant.

Lemma 56. *Let $f \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ and $g \in PAP(\mathbb{R}, \mathbb{R}_+, \mu)$. If $c > 0$, then $fg \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$.*

Proof. According to Theorem 29, there exists a unique $u \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$ such that

$$f(t) \coloneqq c^\wedge(t) u(t), \tag{103}$$

for all $t \in \mathbb{R}$. Using Lemma 13, it is clear that $u \times g \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$. Then,

$$(fg)(t) \coloneqq c^\wedge(t)(u(t)g(t)), \text{ where } ug \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}_+, \mu). \tag{104}$$

Invoking Theorem 29, we complete the proof. □

Now, we can establish following lemma.

Lemma 57. *Let $\mu \in \mathscr{M}_B$. For all $x(\cdot) \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$, the function*

$$\psi_i : t \mapsto \psi_i\left(x_{(\omega,c)}(t)\right) = a(t) e^{-c^\wedge(-t)\omega_i(t) \sum_{j=1}^p x(t-\tau_{ij})} = c^\wedge(t) a_{(\omega,c)}(t) e^{-\omega_i(t) \sum_{j=1}^p c^\wedge(-\tau_{ij}) x_{(\omega,c)}(t-\tau_{ij})} \tag{105}$$

belongs to $PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ for all $1 \le i \le n$.

Proof. First, we can say that the function

$$t \mapsto x(t - \tau_{ij}) \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu), \tag{106}$$

for all $1 \le i \le n$ and $1 \le j \le p$. Then, according to Proposition 30

$$t \mapsto \sum_{j=1}^p x(t - \tau_{ij}) \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu). \tag{107}$$

Furthermore, by Lemma 5,

$$t \mapsto \omega_i(t) \sum_{j=1}^p x(t - \tau_{ij}) \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu). \tag{108}$$

for all $1 \le i \le n$. Now, using the fact that the function ($x \mapsto e^{-x}$) is Lipschitzian and bounded, and $a_{(\omega,c)} \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}_+, \mu)$ is also bounded then invoking the Corollary 54, it is clear that

$$\psi_i : t \mapsto c^{\wedge}(t) a_{(\omega,c)}(t) e^{-c^{\wedge}(-t)\omega_i(t) \sum_{j=1}^{p} x(t-\tau_{ij})} \in \mathrm{PAP}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu), \tag{109}$$

for all $1 \le i \le n$. □

By using condition (C6) and Theorem 37, we can deduce the following Lemma.

Lemma 58. *Suppose that (H1) and (C6) hold, if $x \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$, then the function defined by*

$$t \mapsto \int_{-\infty}^{t} K_j(t-s) x(s) ds \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu), \tag{110}$$

for all $1 \le j \le m$.

Theorem 59. *Suppose that (H1) and (C1) – (C4) are satisfied. Then, the nonlinear operator Γ defined for each $x \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ by*

$$\begin{aligned}(\Gamma x)(t) = \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \Bigg[&\sum_{j=1}^{m} a_j(s) e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s} K_j(s-\sigma)x(\sigma)d\sigma} \\ &+ \sum_{i=1}^{n} b_i(s) e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p} x(s-\tau_{ij})} \Bigg] ds\end{aligned} \tag{111}$$

maps $PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$ into itself.

Proof. Using Lemmas 13, 5, 5, and 5 and Corollary 54, then the function χ defined by

$$\begin{aligned}\chi(s) &= \sum_{j=1}^{m} a_j(s) e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s} K_j(s-\sigma)x(\sigma)d\sigma} + \sum_{i=1}^{n} b_i(s) e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p} x(s-\tau_{ij})} \\ &= c^{\wedge}(s) \Bigg(\sum_{j=1}^{m} (a_j)_{(\omega,c)}(s) e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s} K_j(s-\sigma)x(\sigma)d\sigma} \\ &\quad + \sum_{i=1}^{n} (b_i)_{(\omega,c)}(s) e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p} x(s-\tau_{ij})} \Bigg)\end{aligned} \tag{112}$$

is measure (ω, c)-pseudo-almost periodic.

Consequently, we can write $\chi = \chi_1 + \chi_2$ where $\chi_1 \in AP_{\omega,c}(\mathbb{R}_+)$ and $\chi_2 \in \mathscr{E}_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu)$. It follows that

$$(\Gamma\chi)(t) := \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \chi(s) ds = (\Gamma\chi_1)(t) + (\Gamma\chi_2)(t). \tag{113}$$

Let us show that $(\Gamma\chi_1) \in AP_{\omega,c}(\mathbb{R}_+)$.

We recall that by applying condition (C1) to the model (99), α is almost periodic (i.e., (ω, c)-almost periodic with constant $c = 1$). Now, in the view of the almost periodicity of the function α and the (ω, c)-almost periodicity of the function χ_1, there exists a number l_ε such that in any interval $[\delta, \delta + l_\varepsilon]$ one finds a number η, such that

$$\sup_{\xi\in\mathbb{R}} |\alpha(\xi+\eta) - \alpha(\xi)| < \varepsilon \text{ and } \sup_{\xi\in\mathbb{R}} \left| (\chi_1)_{(\omega,c)}(\xi+\eta) - (\chi_1)_{(\omega,c)}(\xi) \right| < \varepsilon. \tag{114}$$

It comes that

$$\begin{aligned}(\Gamma\chi_1)(t+\eta) - (\Gamma\chi_1)(t) &= \int_{-\infty}^{t+\eta} e^{-\int_s^{t+\eta}\alpha(\xi)d\xi} \chi_1(s) ds \\ &- \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \chi_1(s) ds = \int_{-\infty}^{t+\eta} e^{-\int_{s-\eta}^{t}\alpha(\xi+\eta)d\xi} \chi_1(s) ds \\ &- \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \chi_1(s) ds = \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi+\eta)d\xi} \chi_1(s+\eta) ds \\ &- \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \chi_1(s) ds = \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi+\eta)d\xi} \chi_1(s+\eta) ds \\ &- \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \chi_1(s+\eta) ds + \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \chi_1(s+\eta) ds \\ &- \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \chi_1(s) ds.\end{aligned} \tag{115}$$

So, there exists $\delta \in (0, 1)$ such that

$$\begin{aligned}&\left| (\Gamma\chi_1)_{(\omega,c)}(t+\eta) - (\Gamma\chi_1)_{(\omega,c)}(t) \right| \le |\chi_1|_{(\omega,c)} \int_{-\infty}^{t} \left| e^{-\int_s^t \alpha(\xi+\eta)d\xi} - e^{-\int_s^t \alpha(\xi)d\xi} \right| ds + \\ &\quad \times \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \left| (\chi_1)_{(\omega,c)}(s+\eta) - (\chi_1)_{(\omega,c)}(s) \right| ds \\ &\le |\chi_1|_{(\omega,c)} \int_{-\infty}^{t} \left| e^{-\int_s^t \alpha(\xi+\eta)d\xi} - e^{-\int_s^t \alpha(\xi)d\xi} \right| ds + \varepsilon \\ &\quad \times \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} ds \le |\chi_1|_{(\omega,c)} \int_{-\infty}^{t} \left| e^{-\int_s^t \alpha(\xi+\eta)d\xi} - e^{-\int_s^t \alpha(\xi)d\xi} \right| ds + \varepsilon \\ &\quad \times \int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}} ds \le |\chi_1|_{(\omega,c)} \int_{-\infty}^{t} \\ &\quad \times \left\{ e^{-\left[\int_s^t \alpha(\xi+\eta)d\xi + \delta\left(\int_s^t \alpha(\xi)d\xi - \int_s^t \alpha(\xi+\eta)d\xi\right)\right]} \times \left| \int_s^t \alpha(\xi+\eta) - \alpha(\xi) d\xi \right| ds \right\} + \varepsilon \\ &\quad \times \int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}} ds \le |\chi_1|_{(\omega,c)} \int_{-\infty}^{t} \\ &\quad \times \left\{ e^{-\int_s^t \alpha(\xi+\eta)d\xi} e^{-\delta\left(\int_s^t \alpha(\xi)d\xi - \int_s^t \alpha(\xi+\eta)d\xi\right)} \int_s^t |\alpha(\xi+\eta) - \alpha(\xi)| d\xi ds \right\} \\ &\quad + \varepsilon \int_{-\infty}^{t} e^{-(t-s)\underline{\alpha}} ds,\end{aligned}$$

$$\begin{aligned}&\leq \varepsilon|\chi_1|_{(\omega,c)}\int_{-\infty}^{t}\left[e^{-(t-s)\underline{\alpha}}e^{-\delta\left(\int_s^t\alpha(\xi)d\xi-\int_s^t\alpha(\xi+\eta)d\xi\right)}(t-s)\right]ds+\varepsilon\\&\quad\cdot\int_{-\infty}^{t}e^{-(t-s)\underline{\alpha}}ds\leq\varepsilon|\chi_1|_{(\omega,c)}\int_{-\infty}^{t}\left[e^{-(t-s)\underline{\alpha}}(t-s)\right]ds+\varepsilon\\&\quad\cdot\int_{-\infty}^{t}e^{-(t-s)\underline{\alpha}}ds\leq\frac{\varepsilon|\chi_1|_{(\omega,c)}}{\underline{\alpha}^2}+\varepsilon\int_{-\infty}^{t}e^{-(t-s)\underline{\alpha}}ds\leq\frac{\varepsilon|\chi_1|_{(\omega,c)}}{\underline{\alpha}^2}\\&\quad+\frac{\varepsilon}{\underline{\alpha}}=\left(\frac{|\chi_1|_{(\omega,c)}}{\underline{\alpha}^2}+\frac{1}{\underline{\alpha}}\right)\varepsilon.\end{aligned}\tag{116}$$

This proves that $(\Gamma\chi_1)\in AP_{\omega,c}(\mathbb{R}_+)$. Now, let us show that $(\Gamma\chi_2)\in\mathscr{E}_{(\omega,c)}(\mathbb{R},\mathbb{R}_+,\mu)$. We have that

$$\begin{aligned}&\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left|\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\chi_2(s)ds\right|_{(\omega,c)}d\mu(t)\\&\quad\leq\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{-\infty}^{t}e^{-(t-s)\underline{\alpha}}|\chi_2(s)|_{(\omega,c)}ds\,d\mu(t)\\&\quad\leq\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_{0}^{\infty}e^{-\underline{\alpha}\xi}|\chi_2(t-\xi)|_{(\omega,c)}d\xi\,d\mu(t)\\&\quad=\lim_{r\longrightarrow+\infty}\int_{0}^{\infty}e^{-\underline{\alpha}\xi}\left(\frac{1}{\mu([-r,r])}\int_{-r}^{r}|\chi_2(t-\xi)|_{(\omega,c)}d\mu(t)\right)d\xi.\end{aligned}\tag{117}$$

By the Lebesgue Dominated Convergence Theorem and $(H1)$, we obtain that

$$\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\left|\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\chi_2(s)ds\right|_{(\omega,c)}d\mu(t)=0.\tag{118}$$

Then, $(\Gamma\chi_2)\in\mathscr{E}_{(\omega,c)}(\mathbb{R},\mathbb{R}_+,\mu)$ and consequently, $\Gamma\in PAP_{(\omega,c)}(\mathbb{R},\mathbb{R}_+,\mu)$. □

Theorem 60. *Assume that (H1) and (C1)–(C6) hold, then the Lasota-Wazewska model with mixed delays (99) possesses a unique measure (ω,c)-pseudo-almost periodic solution y, and we have $y_{(\omega,c)}$ in the region*

$$\mathbf{R}=\left\{\psi\in PAP_{(\omega,c)}(\mathbb{R},\mathbb{R}_+,\mu),\mathscr{LB}\leq\left|\psi_{(\omega,c)}\right|\leq\mathscr{UB}\right\},\tag{119}$$

where

$$\begin{aligned}\mathscr{UB}&=\frac{\sum_{j=1}^{m}\overline{(a_j)}_{(\omega,c)}+\sum_{i=1}^{n}\overline{(b_i)}_{(\omega,c)}}{\underline{\alpha}}\text{ and }\mathscr{LB}\\&=\frac{\sum_{j=1}^{m}\underline{(a_j)_{(\omega,c)}}e^{-\bar{\omega}_j\mathscr{UB}}+\sum_{i=1}^{n}\underline{(b_i)_{(\omega,c)}}e^{-p\xi\bar{\beta}_i\mathscr{UB}}}{\bar{\alpha}}.\end{aligned}\tag{120}$$

Proof. First, we proves that the operator Γ is a mapping from $\mathbf{R}$ to $\mathbf{R}$. We set

$$\begin{aligned}\gamma_j(s)&=a_j(s)e^{-c^\wedge(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)x(\sigma)d\sigma},\\\theta_i(s)&=b_i(s)e^{-c^\wedge(-s)\beta_i(s)\sum_{j=1}^{p}x(s-\tau_{ij})}.\end{aligned}\tag{121}$$

In fact, we have

$$\begin{aligned}|\Gamma x(t)|_{(\omega,c)}&\leq\int_{-\infty}^{t}\left|e^{-\int_s^t\alpha(\xi)d\xi}\left[\sum_{j=1}^{m}\gamma_j(s)+\sum_{i=1}^{n}\theta_i(s)\right]\right|_{(\omega,c)}ds\\&=\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\left[\left|\sum_{j=1}^{m}\gamma_j(s)+\sum_{i=1}^{n}\theta_i(s)\right|_{(\omega,c)}\right]ds\\&\leq\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\left[\sum_{j=1}^{m}\overline{(a_j)}_{(\omega,c)}+\sum_{i=1}^{n}\overline{(b_i)}_{(\omega,c)}\right]ds\\&\leq\int_{-\infty}^{t}e^{-\underline{\alpha}(t-s)}\left[\sum_{j=1}^{m}\overline{(a_j)}_{(\omega,c)}+\sum_{i=1}^{n}\overline{(b_i)}_{(\omega,c)}\right]ds\\&=\frac{\sum_{j=1}^{m}\overline{(a_j)}_{(\omega,c)}+\sum_{i=1}^{n}\overline{(b_i)}_{(\omega,c)}}{\underline{\alpha}}.\end{aligned}\tag{122}$$

In the other hand, if we set

$$\begin{aligned}\xi&=\max\left\{c^\wedge(\tau_{i,j})\right\}\text{ for }(i,j)\in[[1,n]]\times[[1,p]],\\\delta_j(s)&=c^\wedge(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)x(\sigma)d\sigma,\text{ for }j\in[[1,m]],\\\phi_i(s)&=c^\wedge(-s)\beta_i(s)\sum_{j=1}^{p}x(s-\tau_{ij})\text{ for }i\in[[1,n]],\end{aligned}\tag{123}$$

then, we have for $x\in\mathbf{R}$

$$\begin{aligned}|\Gamma x(t)|_{(\omega,c)}&=\left|\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\left[\sum_{j=1}^{m}a_j(s)e^{-\delta_j(s)}+\sum_{i=1}^{n}b_i(s)e^{-\phi_i(s)}\right]ds\right|_{(\omega,c)}\geq c^\wedge(-t)\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\\&\quad\cdot\left[\sum_{j=1}^{m}a_j(s)e^{-\bar{\omega}_j\int_{-\infty}^{s}(K_j)_{(\omega,c)}(s-\sigma)x_{(\omega,c)}(\sigma)d\sigma}+\sum_{i=1}^{n}b_i(s)e\right.\\&\quad\left.-\bar{\beta}_i\sum_{j=1}^{p}c^\wedge(-\tau_{ij})x_{(\omega,c)}(s-\tau_{ij})\right])ds\geq c^\wedge(-t)\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\\&\quad\cdot\left[\sum_{j=1}^{m}a_j(s)e^{-\bar{\omega}_j\mathscr{UB}\int_{-\infty}^{s}(K_j)_{(\omega,c)}(s-\sigma)d\sigma}+\sum_{i=1}^{n}b_i(s)e^{-\bar{\beta}_i\mathscr{UB}\sum_{j=1}^{p}c^\wedge(-\tau_{ij})}\right])ds\\&\geq c^\wedge(-t)\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\left[\sum_{j=1}^{m}a_j(s)e^{-\bar{\omega}_j\mathscr{UB}}+\sum_{i=1}^{n}b_i(s)e^{-p\xi\bar{\beta}_i\mathscr{UB}}\right]ds\\&\geq\int_{-\infty}^{t}c^\wedge(-s)e^{-\int_s^t\alpha(\xi)d\xi}\left[\sum_{j=1}^{m}a_j(s)e^{-\bar{\omega}_j\mathscr{UB}}+\sum_{i=1}^{n}b_i(s)e^{-p\xi\bar{\beta}_i\mathscr{UB}}\right]ds\\&=\int_{-\infty}^{t}e^{-\int_s^t\alpha(\xi)d\xi}\left[\sum_{j=1}^{m}(a_j)_{(\omega,c)}(s)e^{-\bar{\omega}_j\mathscr{UB}}+\sum_{i=1}^{n}(b_i)_{(\omega,c)}(s)e^{-p\xi\bar{\beta}_i\mathscr{UB}}\right]ds\\&\geq\int_{-\infty}^{t}e^{-\bar{\alpha}(t-s)}\left[\sum_{j=1}^{m}\underline{(a_j)_{(\omega,c)}}e^{-\bar{\omega}_j\mathscr{UB}}+\sum_{i=1}^{n}\underline{(b_i)_{(\omega,c)}}e^{-p\xi\bar{\beta}_i\mathscr{UB}}\right]ds\\&=\frac{\sum_{j=1}^{m}\underline{(a_j)_{(\omega,c)}}e^{-\bar{\omega}_j\mathscr{UB}}+\sum_{i=1}^{n}\underline{(b_i)_{(\omega,c)}}e^{-p\xi\bar{\beta}_i\mathscr{UB}}}{\bar{\alpha}},\end{aligned}\tag{124}$$

which implies that the operator Γ is a mapping from $\mathbf{R}$ to itself. To end the proof, it suffices to prove that Γ is a contraction mapping. Let $x,y\in\mathbf{R}$. Then,

$$
\begin{aligned}
|\Gamma x(t)-\Gamma y(t)|_{(\omega,c)} = {} & \Bigg|\int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\Bigg[\sum_{j=1}^{m} a_j(s)\left(e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)x(\sigma)d\sigma}-e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)y(\sigma)d\sigma}\right)+\sum_{i=1}^{n} b_i(s) \\
& \cdot\left(e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}x(s-\tau_{ij})}-e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}y(s-\tau_{ij})}\right)\Bigg]ds\Bigg|_{(\omega,c)} \le \int_{-\infty}^{t}\left|e^{-\int_s^t \alpha(\xi)d\xi}\right. \\
& \cdot\Bigg[\sum_{j=1}^{m} a_j(s)\left(e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)x(\sigma)d\sigma}-e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)y(\sigma)d\sigma}\right)+\sum_{i=1}^{n} b_i(s) \\
& \cdot\left(e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}x(s-\tau_{ij})}-e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}y(s-\tau_{ij})}\right)\Bigg]\Bigg|_{(\omega,c)} ds = \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\Big| \\
& \cdot\Bigg[\sum_{j=1}^{m} (a_j)_{(\omega,c)}(s)\left(e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)x(\sigma)d\sigma}-e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)y(\sigma)d\sigma}\right)+\sum_{i=1}^{n} (b_i)_{(\omega,c)}(s) \\
& \cdot\left(e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}x(s-\tau_{ij})}-e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}y(s-\tau_{ij})}\right)\Bigg]\Big|ds \le \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi}\Big| \\
& \cdot\Bigg[\sum_{j=1}^{m} \overline{(a_j)}_{(\omega,c)}\left(e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)x(\sigma)d\sigma}-e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)y(\sigma)d\sigma}\right)+\sum_{i=1}^{n} \overline{(b_i)}_{(\omega,c)} \\
& \cdot\left(e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}x(s-\tau_{ij})}-e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}y(s-\tau_{ij})}\right)\Bigg]\Big|ds \le \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \\
& \cdot\Bigg[\sum_{j=1}^{m} \overline{(a_j)}_{(\omega,c)}\left|e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)x(\sigma)d\sigma}-e^{-c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s}K_j(s-\sigma)y(\sigma)d\sigma}\right|+\sum_{i=1}^{n} \overline{(b_i)}_{(\omega,c)} \\
& \cdot\left|e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}x(s-\tau_{ij})}-e^{-c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}y(s-\tau_{ij})}\right|\Bigg]ds.
\end{aligned}
\tag{125}
$$

Obviously, for $u, v \in [0,+\infty)$,

$$|e^{-u} - e^{-v}| \le |u - v|. \tag{126}$$

Then it comes that

$$\begin{aligned}
|\Gamma x(t) - \Gamma y(t)|_{(\omega,c)} \le \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \\
\Bigg[\sum_{j=1}^{m} \overline{(a_j)}_{(\omega,c)} \left| c^{\wedge}(-s)\omega_j(s)\int_{-\infty}^{s} K_j(s-\sigma)(x(\sigma)-y(\sigma))d\sigma \right| \\
+ \sum_{i=1}^{n} \overline{(b_i)}_{(\omega,c)} \left| c^{\wedge}(-s)\beta_i(s)\sum_{j=1}^{p}\left(x\left(s-\tau_{ij}\right)-y\left(s-\tau_{ij}\right)\right) \right| \Bigg] ds \\
\le \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^{m} \overline{(a_j)}_{(\omega,c)} \omega_j \int_{-\infty}^{s} (K_j)_{(\omega,c)}(s-\sigma)d\sigma |x \right. \\
\left. - y|_{(\omega,c)} + p\xi |x-y|_{(\omega,c)} \sum_{i=1}^{n} \overline{(b_i)}_{(\omega,c)}\beta_i \right] ds \\
= \int_{-\infty}^{t} e^{-\int_s^t \alpha(\xi)d\xi} \left[\sum_{j=1}^{m} \overline{(a_j)}_{(\omega,c)} \omega_j + p\xi \sum_{i=1}^{n} \overline{(b_i)}_{(\omega,c)}\beta_i \right] ds |x \\
- y|_{(\omega,c)} \le \frac{\sum_{j=1}^{m} \overline{(a_j)}_{(\omega,c)} \omega_j + p\xi \sum_{i=1}^{n} \overline{(b_i)}_{(\omega,c)}\beta_i}{\underline{\alpha}} |x-y|_{(\omega,c)},
\end{aligned} \tag{127}$$

which implies (invoking (C5)) that the mapping Γ is a contraction mapping of $\mathbf{R}$. Consequently, Γ possesses a unique fixed point $x^{\star} \in \mathbf{R}$. Hence, $x^{\star}$ is the unique measure (ω, c)-pseudo-almost periodic solution of Equation (99) in $\mathbf{R}$. □

5.2. Example. In order to illustrate some features of our theoretical study, we will apply our main results to a special system and demonstrate the efficiencies of our criteria.

We consider the following Lasota-Wazewska model with mixed delays

$$\begin{aligned}
y'(t) = -\alpha(t)y(t) + \sum_{j=1}^{3} a_j(t) e^{-c^{\wedge}(-t)\omega_j(t)\int_{-\infty}^{t} K_j(t-s)y(s)ds} \\
+ \sum_{i=1}^{3} b_i(t) e^{-c^{\wedge}(-t)\beta_i(t)y(t-\tau_i)},
\end{aligned} \tag{128}$$

where $\alpha(t) = 8 + \cos^2(\sqrt{5}t) + \cos^2(t)$, $c = 0.9$, and $\omega = 1$,

$$\begin{aligned}
\begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{pmatrix} &= \begin{pmatrix} (0.9)^{\wedge}(t)\left(1 + 0.25\cos^2\left(\sqrt{2}t\right) + 0.25\cos^2(\pi t) + \frac{0.5}{1+t^2}\right) \\ (0.9)^{\wedge}(t)\left(0.5 + 0.25\cos^2\left(\sqrt{3}t\right) + 0.25\cos^2(\pi t) + \frac{1}{1+t^2}\right) \\ (0.9)^{\wedge}(t)\left(0.5 + 0.25\cos^2\left(\sqrt{5}t\right) + 0.25\cos^2\left(\sqrt{2}t\right) + e^{-t^2\cos^2(t)}\right) \end{pmatrix}, \\
\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{pmatrix} &= \begin{pmatrix} 0.125\cos^2\left(\sqrt{2}t\right) + 0.125\cos^2(\pi t) + \frac{0.25}{1+t^2} \\ 0.125\cos^2\left(\sqrt{2}t\right) + 0.125\cos^2(\pi t) + \frac{0.25}{1+t^2} \\ 0.250\cos^2\left(\sqrt{2}t\right) + 0.25e^{-t^2\cos^2(t)} \end{pmatrix}, \\
\begin{pmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix} &= \begin{pmatrix} (0.9)^{\wedge}(t)\left(1 + 0.25\cos^2\left(\sqrt{5}t\right) + 0.25\cos^2(\pi t) + 0.5e^{-t^2\cos^2(t)}\right) \\ (0.9)^{\wedge}(t)\left(1 + 0.25\cos^2\left(\sqrt{5}t\right) + 0.25\cos^2(\pi t) + 0.5e^{-t^2\cos^2(t)}\right) \\ (0.9)^{\wedge}(t)\left(1 + 0.25\cos^2\left(\sqrt{5}t\right) + 0.25\cos^2(\pi t) + 0.5e^{-t^2\cos^2(t)}\right) \end{pmatrix}, \\
\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{pmatrix} &= \begin{pmatrix} 0.125\cos^2\left(\sqrt{2}t\right) + 0.125\cos^2(\pi t) + \frac{0.25}{1+t^2} \\ 0.125\cos^2\left(\sqrt{2}t\right) + 0.125\cos^2(\pi t) + \frac{0.25}{1+t^2} \\ 0.125\cos^2\left(\sqrt{2}t\right) + 0.125\cos^2(\pi t) + 0.25e^{-t^2\cos^2(t)} \end{pmatrix}.
\end{aligned} \tag{129}$$

$\tau_1 = \tau_2 = \tau_3 = 1$, $K_j = (0.9)^{\wedge}(t)e^{-t}$. Then,

$$\frac{\sum_{j=1}^{m} \overline{(a_j)}_{(\omega,c)} \omega_j + p\xi \sum_{i=1}^{n} \overline{(b_i)}_{(\omega,c)}\beta_i}{\underline{\alpha}} \le \frac{3}{4} < 1. \tag{130}$$

If the Radon-Nikodym derivative ρ of the measure μ is $\rho(t) = e^{\sin t}$ with respect to the Lebesgue measure on $\mathbb{R}$ (i.e. $d\mu = \rho(t)dt$), then $\mu \in \mathscr{M}_{\mathrm{B}}$, since

$$\mu([-r, r]) = \int_{-r}^{r} e^{\sin(t)} \longrightarrow +\infty, \text{if } r \longrightarrow +\infty, \tag{131}$$

$$\mu(\tau + a) \le e^2\mu(A), \forall \tau \in \mathbb{R} \text{ and } a \in A. \tag{132}$$

Hence, conditions (C1) – (C6) and $(H1)$ are satisfied then according to the Theorem 60, the Lasota-Wazewska model with a mixed delays (14) has a unique μ-(ω, c)-pseudo-almost periodic solution in the region $\mathbf{R} = \{y \in PAP_{(\omega,c)}(\mathbb{R}, \mathbb{R}_+, \mu), \mathscr{LB} \le |y_{(\omega,c)}| \le \mathscr{UB}\}$ where

$$\mathscr{UB} = \frac{\sum_{j=1}^{m} \overline{(a_j)}_{(\omega,c)} + \sum_{i=1}^{n} \overline{(b_i)}_{(\omega,c)}}{\underline{\alpha}} = \frac{3}{2}, \tag{133}$$

$$\begin{aligned}
\mathscr{LB} &= \frac{\sum_{j=1}^{m} \underline{\left(a_j\right)}_{(\omega,c)} e^{-\bar{\omega}_j \mathscr{UB}} + \sum_{i=1}^{n} \underline{\left(b_i\right)}_{(\omega,c)} e^{-p\xi\bar{\beta}_i \mathscr{UB}}}{\bar{\alpha}} \\
&\le \frac{e^{-1/23/2} + 0.5e^{-1/23/2} + 0.5e^{-1/23/2} + e^{-0.91/23/2} + e^{-0.91/23/2} + e^{-0.91/23/2}}{10} \\
&= \frac{e^{-3/4}}{5} + \frac{3e^{-0.9/4}}{10} \approx 0.246.
\end{aligned} \tag{134}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest and no data used.

References

[1] H. Bohr, *Almost Periodic Functions*, Chelsea Publishing Co., New York, 1947.

[2] A. Alvarez, A. Gómez, and M. Pinto, "(ω,c)-periodic functions and mild solutions to abstract fractional integro-differential equations," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 9, no. 16, pp. 1–8, 2018.

[3] L. Abadias, E. Alvarez, and R. Grau, "(ω, c)-periodic mild solutions to non-autonomous abstract differential equations," *Mathematics*, vol. 474, 2021.

[4] J. Blot, P. Cieutat, and K. Ezzinbi, "New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications," *Applicable Analysis*, vol. 92, no. 3, pp. 493–526, 2013.

[5] C. Corduneanu, *Almost Periodic Functions*, Wiley, New York, 1968, (Reprinted, Chelsea, New York, 1989).

[6] T. Diagana, "Les fonctions pseudo presque periodiques avec poids et applications," *Comptes Rendus Mathematique*, vol. 343, no. 10, pp. 643–646, 2006.

[7] R. G. Foko Tiomela and G. M. N'Guérékata, "(ω,c)-almost periodic solutions to some fractional integrodifferenial equation," *Journal of Fractional Calculus and Applications*, vol. 13, no. 2, p. 199, 2022.

[8] M. Hasler and G. M. N'Guérékata, "Bloch-periodic fucntions and some applications," *Nonlinear Studies*, vol. 21, pp. 21–30, 2014.

[9] M. Kéré, G. M. N'Guérékata, and E. R. Oueama-Guengai, "An existence result of (ω,c)-almost periodic mild solution to some fractional differential equations," *Panamerican Mathematical Journal*, vol. 31, no. 3, pp. 11–20, 2021.

[10] G. M. N'Guérékata, *Almost periodic and almost atomorphic functions in abstract spaces*, Springer, New York, 2nd edition, 2021.

[11] C. Zhang, "Pseudo almost periodic solutions of some differential equations," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 1, pp. 62–76, 1994.

[12] C. Zhang, "Pseudo almost periodic solutions of some differential equations, II," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 2, pp. 543–561, 1995.

[13] T. Diagana, "Weighted pseudo-almost periodic solutions to some differential equations," *Nonlinear Analysis: Theory Methods & Applications*, vol. 68, no. 8, pp. 2250–2260, 2008.

[14] M. T. Khalladi, M. Kostić, A. Rahmani, and D. Velinov, "(ω, c)-Pseudo almost periodic functions, (ω,c)- pseudo almost automorphic functions and applications," *Facta Universitatis. Series: Mathematics and Informatics*, vol. 36, no. 1, pp. 165–177, 2021.

[15] C. Zhang, "Integration of vector-valued pseudo-almost periodic functions," *Proceedings of the American Mathematical Society*, vol. 121, no. 1, pp. 167–174, 1994.

[16] J. Larrouy and G. M. N'Guérékata, "(ω,c)-periodic and asymptotically (ω,c)-periodic mild solutions to fractional Cauchy problems," *Applicable Analysis*, pp. 1–19, 2021.

[17] M. T. Khalladi, M. Kostić, A. Rahmani, and D. Velinov, "(ω,c)-Almost periodic type functions and applications," 2020, http://arxiv.org/abs/0254.9066.

[18] M. Kostić, "Multi-dimensional (ω,c)-almost periodic type functions and applications," *Nonautonomous Dynamical Systems*, vol. 8, no. 1, pp. 136–151, 2021.

[19] L. Schwartz, *Topologie GÃ©nÃ© Rale et Analyse Fonctionnelle*, Hermann, Paris, 1976, (in French).

[20] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, vol. 14 of Applied Mathematical Sciences, Springer-Verlag, New York, 1975.

[21] P. Cieutat, S. Fatajou, and G. M. N'Guérékata, "Composition of pseudo-almost periodic and pseudo-almost automorphic functions and applications to evolution equations," *Applicable Analysis*, vol. 89, no. 1, pp. 11–27, 2010.

[22] M. Wazewska-Czyzewska and A. Lasota, "Mathematical problems of the dynamics of the red blood cells systems," *Annals of the Polish Mathematical Society Series III, Applied Mathematics*, vol. 6, pp. 23–40, 1976.

[23] K. Gopalsamy and S. Trofimchuk, "Almost periodic solutions of Lasota-Wazewska-type delay differential equation," *Journal of Mathematical Analysis and Applications*, vol. 237, no. 1, pp. 106–127, 1999.

[24] F. Cherif and M. Miraoui, "New results for a Lasota-Wazewska model," *International Journal of Biomathematics*, vol. 12, no. 2, article 1950019, 2019.