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Research Article

On Nonsmooth Global Implicit Function Theorems for Locally Lipschitz Functions from Banach Spaces to Euclidean Spaces

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In this paper, we establish a generalization of the Galewski-Rădulescu nonsmooth global implicit function theorem to locally Lipschitz functions defined from infinite dimensional Banach spaces into Euclidean spaces. Moreover, we derive, under suitable conditions, a series of results on the existence, uniqueness, and possible continuity of global implicit functions that parametrize the set of zeros of locally Lipschitz functions. Our methods rely on a nonsmooth critical point theory based on a generalization of the Ekeland variational principle.

1. Introduction

Many mathematical models involving real or vector-valued functions stand as equations of the form

$$f(x) = 0. \tag{1}$$

For complex phenomena, the unknown x is often a vector-variable $x = (x_1, x_2, \cdots, x_n)$ belonging to $\mathbb{R}^n$ or to an abstract Banach space having a direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_n$. It may even happen that equation (1) is just a state equation depending in fact on a parameter (or a control) h. In this case, it takes the form

$$F(x, h) = 0, \tag{2}$$

and the most aspiring aim of mathematical analysis is to know the local or global structure of the solution set $F^{-1}(0)$ by finding out whether it is nonempty, discrete, a graph or a manifold, etc.

The essence of the implicit function theorem in mathematical analysis is to ascertain if the solutions to an equation involving parameters exist and may be viewed locally as a function of those parameters and to know a priori which properties this function might inherit from those of the data. Geometrically, implicit function theorems provide sufficient conditions under which the solution set in some neighborhood of a given solution is the graph of some function. The well-known implicit function theorems deal with a continuous differentiability hypothesis and in such cases are equivalent to inverse function theorems (see [1]). It was originally conceived (in the complex variable form in a pioneering work by Lagrange) over two centuries ago to tackle celestial mechanics problems. Subsequently, it attracted Cauchy who managed to provide its rigorous version and became its discoverer. Later, the generalization of this implicit function theorem to the case of finitely many real variables was proved for the first time by Dini. In this way, the classical theory of implicit functions started with single variables and have progressed through multiple real variables to equations in infinite dimensional spaces, e.g., functional equations involving integral or differential operators. Nowadays, most categories of smooth functions have virtually their own version of the implicit function theorem, and there are special versions adapted to Banach spaces and algebraic geometry and to various types of geometrically degenerate situations. Some of these (such as Nash-Moser implicit function theorem) are quite sophisticated and have been used in amazing ways to solve important open problems (in Riemannian manifolds, partial differential equations, functional analysis, …) [1]. There are also in the literature [2, 3] some implicit numerical schemes used to approximate the solutions of certain differential equations and that could be regarded as implicit functions in sequence spaces.

Nevertheless, there are interesting phenomena governed by parametric equations with nonsmooth data which need to be stressed and are more and more attracting researchers. Indeed, the implicit function theorems for nondifferentiable functions are less known but are regaining interest in the literature due to their importance in applied sciences that deal with functions having less regularity than smoothness. Few versions have been stated in Euclidean spaces for functions that are continuous with respect to all their variables and (partially) monotone with respect to some of their variables [4, 5].

Recently, Galewski and Rădulescu [6] proved a generalized global implicit function theorem for locally Lipschitz function $F : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$, by using a nonsmooth Palais-Smale condition and a coercivity condition. Their proof is essentially based on the fact that a locally Lipschitz function in a finite dimension is almost everywhere differentiable with respect to the Lebesgue measure according to Rademacher's theorem [7]. It is known that Rademacher's theorem for locally Lipschitz functions has no direct infinite dimensional extension. This justifies all difficulties to have conditions of existence of local or global implicit function in the case of locally Lipschitz function defined on infinite dimensional space (see [8]). Several works have been done to overcome these difficulties. For example, the papers [9, 10] provided conditions for surjectivity and inversion of locally Lipschitz functions between Banach spaces under assumptions formulated in terms of pseudo-Jacobian.

In this work, our aim is to establish under suitable conditions a global implicit function theorem for locally Lipschitz map $F : X \times Y \longrightarrow H$, where X, Y are real Banach spaces and H is a real Euclidean space, and to provide conditions under which this implicit function is continuous. This extends Theorem 30 of Galewski and Rădulescu to the locally Lipschitz functions in infinite dimension with a very relatively simple method compared to those used for this purpose. Knowing that there exist noncoercive functions satisfying the (h)-condition (see Definition 18 and Remark 19), we work in this paper under the (h)-condition using a variational approach and applying a recent nonsmooth version of Mountain Pass Theorem, namely, Theorem 27.

The contribution of this work is quadruple:

(i) An improvement of the classical Clarke's implicit function Theorem 24 for function $F : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ by replacing $\mathbb{R}^p$ by any Banach space Y (Remark 26). Consequently, by considering the approach used in [6] (Theorem 4) and Remark 26, we prove our first main result (Theorem 31) on the existence and uniqueness of global implicit function theorem for equation $F(x, y) = 0$, where $F : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^n$ with Y a Banach space

(ii) The proof of the continuity of the implicit function based on a simple additional hypothesis, Theorem 35

(iii) The weakening of the coercivity assumption used in [6] by considering a compactness type condition called (h)-condition in [11]

(iv) By our Lemmas 42 and 43, we obtain Theorem 38 on the existence and uniqueness of global implicit functions under the (h)-condition on the function $x \mapsto \|F(x, y)\|^\alpha$ with $0 < \alpha < 2$. This is a generalization of the result (49) in the nonsmooth case. It also generalizes the result [12] (Theorem 3.6) in the C^1 case

This article is organized as follows. In Section 2, we recall some preliminary and auxilliary results on Clarke's generalized gradient, Clarke's generalized Jacobian, and the (h)-condition for locally Lipschitz functions. Section 3 is devoted to our main results established under the (h)-condition, on the existence and uniqueness of global implicit function for equation $F(x, y) = 0$, where F is defined from $\mathbb{R}^n \times Y$ to $\mathbb{R}^n$ and Y is a Banach space, namely, Theorems 31, 35, 38, 39, and 40. In Section 4, we give an example of a function satisfying our conditions of existence of implicit function but not the conditions of Theorem 1 of 6 which we have extended. This is the energy functional defined in (139), of a certain differential inclusion problem involving the p-Laplacian [13].

2. Preliminaries and Auxilliary Results

Let U be a nonempty open subset of a Banach space X and let $f : U \longrightarrow \mathbb{R}$ be a function. We recall that f is Lipschitz if there exists some constant $K > 0$ such that for all y and z in U, we have

$$|f(y) - f(z)| \le K\|y - z\|. \tag{3}$$

For $x \in U$, f is said to be locally Lipschitz at x if there exists an open neighborhood $V \subset U$ of x on which the restriction of f is Lipschitz. We will say that f is locally Lipschitz on U if f is locally Lipschitz at every point $x \in U$. We recall that any convex function has this property in Euclidean spaces.

Definition 1. Let $f : U \subset X \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Let $x \in U$ and $v \in X \setminus \{0\}$. The generalized directional derivative of f at x in the direction v, denoted by $f^0(x\,;v)$, is defined by

$$f^0(x\,;v) \coloneqq \limsup_{\substack{w \longrightarrow x \\ t \longrightarrow 0^+}} \frac{f(w + tv) - f(w)}{t}. \tag{4}$$

Observe at once that $f^0(x\,;v)$ is a (finite) number for all $v \in X \setminus \{0\}$.

Indeed, let $x \in V \subset U$ and let $K > 0$ be such that (3) holds for all $y, z \in V$, with V bounded (without loss of generality). Let $(w_m)_{m>0} \subset X$ be a sequence such that $w_m \longrightarrow x$ and t_m a sequence of $(0\,;+\infty)$ such that $t_m \longrightarrow 0$. For $v \in X \setminus \{0\}$, as $m \longrightarrow +\infty$, the vectors $w_m + t_m v$ will belong to V. Indeed, by boundedness of V, there exists $\rho > 0$ such that $\|x - y\| < \rho \Rightarrow y \in V$. Then, for m large enough, we have

$$\|(w_m + t_m v) - x\| \le \|w_m - x\| + t_m\|v\| < \frac{\rho}{2} + \frac{\rho}{2} = \rho. \tag{5}$$

Thus, there exists $m_0 > 0$ such that for all $m > m_0$, we have

$$\frac{|f(w_m + t_m v) - f(w_m)|}{t_m} \leq K\|v\|. \tag{6}$$

It follows from (3) and (6) that for all $v \in X$,

$$\left|f^0(x, v)\right| \leq K\|v\|. \tag{7}$$

Remark 2. If f is locally Lipschitz and Gâteaux differentiable at x, then its Gâteaux differential $f'_G(x)$ at x coincides with its generalized gradient. That is,

$$f^0(x\,;v) = f'_G(x) \cdot v \text{ for all } v \in X. \tag{8}$$

Proposition 3. *The function $v \mapsto f^0(x\,;v)$ is positively homogeneous and subadditive.*

Proof. The homogeneity is an immediate consequence of Definition 1. We prove the subadditivity. Let v and z be in X. Then,

$$\begin{aligned} f^0(x\,;v+z) &= \limsup_{\substack{w\longrightarrow x\\ t\longrightarrow 0^+}} \frac{f(w+tv+tz)-f(w)}{t} \\ &\leq \limsup_{\substack{w\longrightarrow x\\ t\longrightarrow 0^+}} \frac{f(w+tz+tv)-f(w+tz)}{t} \\ &\quad + \limsup_{\substack{w\longrightarrow x\\ t\longrightarrow 0^+}} \frac{f(w+tz)-f(w)}{t} \\ &\leq \limsup_{\substack{r\longrightarrow x\\ t\longrightarrow 0^+}} \frac{f(r+tv)-f(r)}{t} \\ &\quad + \limsup_{\substack{w\longrightarrow x\\ t\longrightarrow 0^+}} \frac{f(w+tz)-f(w)}{t}, r := w+tz \\ &= f^0(x\,;v) + f^0(x\,;z). \end{aligned} \tag{9}$$

□

From the previous Proposition 3 and the Hahn-Banach theorem [14] (p. 62), it follows that there exists at least one linear function $\xi^* : X \longrightarrow \mathbb{R}$ satisfying

$$f^0(x\,;v) \geq \langle \xi^*, v\rangle \tag{10}$$

for all $v \in X$. From (10) and (7) also rewritten with $(-v)$, we obtain

$$\left|\langle \xi^*, v\rangle\right| \leq K\|v\| \tag{11}$$

for all $v \in X$. Thus, $\xi^* \in X^*$ (as usual, X^* denotes the (continuous) dual of X and <.,.> is the duality pairing between X and X^*). Thus, we can give the following definition.

Definition 4. Let $f : U \subset X \longrightarrow \mathbb{R}$ be locally Lipschitz at a point $x \in U$. Clarke's generalized gradient of f at x, denoted $\partial f(x)$, is the (nonempty) set of all $\xi^* \in X^*$ satisfying (10), i.e.,

$$\partial f(x)s := \left\{\xi^* \in X^* : \forall v \in X, f^0(x\,;v) \geq \langle \xi^*, v\rangle\right\}. \tag{12}$$

We refer to [15–17] for some of the fundamental results in the calculus of generalized gradients. In particular, we shall need the following.

Proposition 5 (see [18], Chang). *If $f : U \longrightarrow \mathbb{R}$ is a convex function, then Clarke's generalized gradient of f at x, defined in (12), coincides with the subdifferential of f in the sense of convex analysis.*

Proposition 6 (see [11], Chen). *Let X be a real Banach space and $f : X \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Then, the function $\gamma : X \longrightarrow \mathbb{R}$ defined by*

$$\gamma(u) := \min_{x^*\in\partial f(u)} \|x^*\|, \textit{for all } u \in X, \tag{13}$$

is well defined and lower semicontinuous.

Proposition 7 (see [15], Proposition 6). *If x_0 is a minimizer of f, then $0 \in \partial f(x_0)$.*

Remark 8. Let X be an infinite dimensional Banach space and $f : X \longrightarrow \mathbb{R}^p$ be a locally Lipschitz mapping. For any finite dimensional subspace of X, it makes sense to talk about Clarke's generalized Jacobian of the function $f_L : L \ni x \mapsto f(x) \in \mathbb{R}^p$ at every point $x \in L$.

Notation 9. . For a locally Lipschitz function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $x \in \mathbb{R}^n$, we consider the set $\Omega_f(x)$ defined by $\Omega_f(x) := \{(x_m)_m$ sequence in $\mathbb{R}^n$ such that $x_m \longrightarrow x$ and f is differentiable at $x_m\}$.

Let X, Z be two Banach spaces such that $\dim Z = n < \infty$. Let $F : X \longrightarrow Z$ be a locally Lipschitz mapping and L a finite dimensional subspace of X. For $x \in L$, we denote by $\partial F_L(x)$ Clarke's generalized Jacobian at a point x, of the restriction of F to L, namely, the function

$$F_L : L \longrightarrow Z\,; x \mapsto F(x). \tag{14}$$

Let Y be a Banach space and consider a function $F : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^p$ which is locally Lipschitz. For any $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times Y$, $\partial_x F(\bar{x}, \bar{y})$ denotes Clarke's generalized Jacobian at a point $\bar{x}$ of the function

$$F(\cdot, \bar{y}) \colon \mathbb{R}^n \longrightarrow \mathbb{R}^p, x \mapsto F(x, \bar{y}). \tag{15}$$

Let X, Y, Z be three Banach spaces with $\dim Z < \infty$ and

$F : X \times Y \longrightarrow Z$ a locally Lipschitz function. For any finite dimensional subspace L of X and for every $(\bar{x}, \bar{y}) \in L \times Y$, $\partial_x F_L((\bar{x}, \bar{y})$ will denote Clarke's generalized Jacobian of the function $\tilde{F} : L \ni x \mapsto \tilde{F}(x) \coloneqq F(x, \bar{y}) \in Z$ at a point $\bar{x}$.

Theorem 10 (Rademacher). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Then, f is almost everywhere differentiable with respect to Lebesgue measure.*

According to Rademacher's Theorem 10, we have the following.

Proposition 11 (see [19], Clarke). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz function and $x \in \mathbb{R}^n$. If $\partial f(x)$ denotes the set defined by (12), then*

$$\partial f(x) = co\left\{ \lim_{m \longrightarrow +\infty} f'(x_m) : (x_m)_{m\in\mathbb{N}} \in \Omega_f(x) \right\}. \tag{16}$$

Note that, since f is almost everywhere differentiable with respect to Lebesgue measure, there exists a sequence $(x_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n$ such that $x_m \longrightarrow x$, and for any $m \in \mathbb{N}$, f is differentiable at x_m. So, $\Omega_f(x) \neq \emptyset$. In addition for any $(x_m)_{m\in\mathbb{N}} \in \Omega_f(x)$ and for any $v \in \mathbb{R}^n$, we have

$$\left| f'(x_m) \cdot v \right| \leq K|v|, \tag{17}$$

where K is the Lipschitz constant of f. This means that $(f'(x_m))_m$ is bounded in $\mathscr{L}(\mathbb{R}^n, \mathbb{R})$ which has a finite dimension. Then, there exists a subsequence $(f'(x_{\sigma(m)}))_m$ of $(f'(x_m))_m$ that converges to some $x^* \in \mathscr{L}(\mathbb{R}^n, \mathbb{R})$. That is,

$$\lim_{m \longrightarrow +\infty} f'\left(x_{\sigma(m)}\right) = x^*. \tag{18}$$

Thus, the convex hull of such limits in (18) is $\partial f(x)$.

Even if the function f is defined from $\mathbb{R}^n$ to $\mathbb{R}^p$, regarding (17) and (18) component by component, we notice that the set defined by (16) is nonempty, compact, and convex in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^p)$ (see [20] (Definition 1)). Thus, this characterization of $\partial f(x)$ stated in Proposition 11 is extended to locally Lipschitz functions defined from $\mathbb{R}^n$ to $\mathbb{R}^p$. In this case, $\partial f(x)$ is called Clarke's generalized Jacobian of the function f at a point x.

Definition 12. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a locally Lipschitz mapping and $x \in \mathbb{R}^n$. Clarke's generalized Jacobian of f at x also denoted by $\partial f(x)$ is defined as follows:

$$\partial f(x) = co\left\{ \lim_{m \longrightarrow +\infty} f'(x_m) : (x_m)_{m\in\mathbb{N}} \in \Omega_f(x) \right\}. \tag{19}$$

The following notions will also be useful in the sequel.

Definition 13. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a locally Lipschitz mapping and $x \in \mathbb{R}^n$ with $n \geq p$. We say that $\partial f(x)$ is of maximal rank if for all $x^* \in \partial f(x)$, x^* is surjective.

Definition 14. Let X be a metric space. A function $f : X \longrightarrow \mathbb{R}$ is said to be (sequentially) lower semicontinuous at a point $x \in X$, if for all sequence $(x_m)_{m\in\mathbb{N}} \subset X$ such that $x_m \longrightarrow x$, we have the inequality

$$f(x) \leq \liminf_{m \longrightarrow +\infty} f(x_m). \tag{20}$$

If for all sequence $(x_m)_{m\in\mathbb{N}} \subset X$ such that $x_m \rightharpoonup x$, (20) holds; we say that f is weakly sequentially lower semicontinuous at x.

Remark 15. Let X be a normed vector space and $(x_m)_m$ a sequence of X. If $x \in X$, then

$$x_m \longrightarrow x \Rightarrow x_m \rightharpoonup x. \tag{21}$$

It follows that the weakly sequentially lower semicontinuity implies the sequentially lower semicontinuity. But the converse is not generally true. However, in the convex case, these two notions are equivalents.

The following theorem is a generalization of Ekeland's variational principle [21].

Theorem 16 (see [21], J. Chen). *Let $h : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous nondecreasing function such that*

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty. \tag{22}$$

Let M be a complete metric space, $x_0 \in M$ fixed, $f : M \longrightarrow \mathbb{R} \cup \{\infty\}$ a lower semicontinuous function, not identically $+\infty$, and bounded from below. Then, for every $\varepsilon > 0$, and $y \in M$ such that

$$f(y) < \inf_M f + \varepsilon, \tag{23}$$

and every $\lambda > 0$, there exists some point $z \in M$ such that

$$\begin{gathered} f(z) < f(y), d(z, x_0) \leq r_0 + \bar{r}, \\ f(x) \geq f(z) - \frac{\varepsilon}{\lambda[1 + h(d(x_0, z))]} d(x, z), \forall x \in M, \end{gathered} \tag{24}$$

where $r_0 = d(x_0, y)$ and $\bar{r}$ is such that

$$\int_{r_0}^{r_0 + \bar{r}} \frac{ds}{1 + h(s)} \geq \lambda. \tag{25}$$

By Theorem 16, one has the following.

Theorem 17 (see [21], J. Chen). *Let X be a Banach space, $h : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous nondecreasing function such that*

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty \tag{26}$$

and $f : X \longrightarrow \mathbb{R}$ a locally Lipschitz function, bounded *from below. Then, there exists a minimizing sequence* $(z_m)_m$ *of* f *such that*

$$f^0(z_m; v - z_m)(1 + h(\|z_m\|)) \geq -\varepsilon_m \|v - z_m\|, \forall v \in X, \quad (27)$$

where $\varepsilon_m \longrightarrow 0^+$ *as* $m \longrightarrow +\infty$.

Proof. For each positive integer m, choose $y_m \in Y$ be such that

$$f(y_m) \leq \inf_M f + \varepsilon_m. \quad (28)$$

□

Take $x_0 = 0, X = M$, and $\lambda = 1$ in Theorem 16. Then, there exists $z_m \in X$ such that

$$\begin{aligned} &f(z_m) \leq f(y_m), \|z_m\| \leq \|y_m\| + \bar{r}, \\ &f(x) \geq f(z_m) - \frac{\varepsilon_m}{[1 + h(\|z_m\|)]} \|x - z_m\|, \forall x \in X, \end{aligned} \quad (29)$$

where $\bar{r}$ is such that

$$\int_{\|y_m\|}^{\|y_m\|+\bar{r}} \frac{ds}{1 + h(s)} \geq 1. \quad (30)$$

Consequently, for each $x \in X$, one has

$$\begin{aligned} &\inf_{\substack{\varepsilon>0 \\ \delta>0}} \sup_{\substack{\|w\|<\varepsilon \\ 0<t<\delta}} \frac{f(z_m + w - t(x - z_m)) - f(z_m + w)}{t} \\ &= \inf_{\delta>0} \sup_{0<t<\delta} \frac{f(z_m + t(x - z_m)) - f(z_m)}{t} \geq \frac{-\varepsilon_m \|x - z_m\|}{1 + h(\|z\|_m)}. \end{aligned} \quad (31)$$

Hence, $f^0(z_m; v - z_m)(1 + h(\|z_m\|)) \geq -\varepsilon_m \|v - z_m\|$, for all $v \in X$.

Moreover, obviously, $(z_m)_m$ is a minimizing sequence of f.

Definition 18. Let X be a Banach space, $f : X \longrightarrow \mathbb{R}$ be bounded from below, locally Lipschitz function, and $h : [0, +\infty) \longrightarrow [0, +\infty)$ be continuous nondecreasing function such that

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty. \quad (32)$$

We say that $(u_m)_{n\geq 0} \subset X$ is a (h)-sequence of f if $(f(u_m))_m$ is bounded and $f^0(u_m; v - u_m)(1 + h(\|u_m\|)) \geq -\varepsilon_m \|v - u_m\|$, for all $v \in X$, where $\varepsilon_m \longrightarrow 0^+$. We say that f satisfies the (h)-condition if any (h)-sequence of f possesses a convergent subsequence.

Remark 19. Sometimes, the following version of (h)-condition is also used: Any sequence $(u_m)_m \subset X$ such that $(f(u_m))_m$ is bounded and

$$\lim_{m \longrightarrow \infty} \gamma(u_m)(1 + h(\|u_m\|)) = 0 \quad (33)$$

possesses a convergent subsequence, where γ is defined in Proposition 6. This condition is equivalent to that of Definition 18.

Remark 20. A coercive function defined on $\mathbb{R}^n$ satisfies the (h)-condition regardless of h. But a function satisfying the (h)-condition is not necessary coercive. Indeed, Section 4 is devoted to the exposition of an example of a noncoercive function satisfying the (h)-condition. It is the function defined in (139).

The following is the Weierstrass theorem.

Lemma 21 (see [13], Lemma 2.1). *Assume that* $f : X \longrightarrow \mathbb{R}$ *is functional on a reflexive Banach space* X *which is weakly lower semicontinuous and coercive. Then, there exists* $x^* \in X$ *such that* $f(x^*) = \min_{x\in X} f(x)$.

Better, by virtue of Theorem 17, we can prove the following result.

Theorem 22. *Let* X *be a Banach space,* $h : [0, +\infty) \longrightarrow [0, +\infty)$ *a continuous nondecreasing function such that*

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty, \quad (34)$$

and $f : X \longrightarrow \mathbb{R}$ *a locally Lipschitz function and bounded from below. If* f *satisfies the* (h)*-condition, then* f *achieves its minimum at some critical point* $z \in X$ *of* f.

Proof. By virtue of Theorem 17, there exists a minimizing sequence $(z_m)_m$ of f and

$$f^0(z_m; v - z_m)(1 + h(\|z_m\|)) \geq -\varepsilon_m \|v - z_m\| \text{ for all } v \in X \quad (35)$$

where $\varepsilon_m \longrightarrow 0^+$. Since f satisfies the (h)-condition, $(z_m)_m$ has a convergent subsequence in X. We can assume that $z_m \longrightarrow z$ in X. Consequently, by the continuity of f,

$$f(z) = \lim_{m \longrightarrow +\infty} f(z_m) = \inf_{x\in X} f(x). \quad (36)$$

□

By Remark 19 and the lower continuity of γ, we know $\gamma(z) = 0$.

Theorem 23 (see [22], Clarke). *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a locally Lipschitz mapping such that the Clarke generalized Jacobian* $\partial f(x_0)$ *of* f *at a point* $x_0 \in \mathbb{R}^n$ *is of maximal rank. Then, there exist neighborhoods* U *and* V *of* x_0 *and* $f(x_0)$*, respectively, and a Lipschitz function* $g : V \longrightarrow U$ *such that* $f(g(u)) = u$ *for all* $u \in U$ *and* $g(f(u)) = v$ *for all* $v \in V$.

The following result is Clarke's implicit function theorem which will be very useful.

Theorem 24 (see [6], Clarke]. *Assume that* $F : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ *is a locally Lipschitz mapping on a neighborhood of a point* (x_0, y_0) *such that* $F(x_0, y_0) = 0$*. Assume further that* $\partial_x F(x_0, y_0)$ *is of maximal rank. Then there exists a neighborhood* $V \subset \mathbb{R}^p$ *of* y_0 *and a Lipschitz function* $G : V \longrightarrow \mathbb{R}^n$ *such that for every* y *in* V*, it holds*

$$\begin{aligned} F(G(y), y) &= 0, \\ G(y_0) &= x_0. \end{aligned} \tag{37}$$

Remark 25. It would be important to point out that the Clarke implicit function Theorem 24 is a corollary of the Clarke inverse function Theorem 23 that can be found in the book [23]. Indeed, as it is done for example in [24] on page 256, when we put

$$\begin{aligned} \tilde{F} : \mathbb{R}^n \times \mathbb{R}^p &\longrightarrow \mathbb{R}^n \times \mathbb{R}^p, \\ (x, y) &\mapsto (F(x, y), y). \end{aligned} \tag{38}$$

$\tilde{F}$ is locally Lipschitz in a neighborhood of (x_0, y_0). Moreover, when the Jacobian matrix $D\tilde{F}$ exists, it is of the form

$$\begin{pmatrix} D_x F & D_y F \\ 0_n & I_p \end{pmatrix}, \tag{39}$$

and it follows that the Clarke generalized Jacobian $\partial\tilde{F}(x_0, y_0)$ of $\tilde{F}$ at the point (x_0, y_0) is of maximal rank. Then, by Theorem 4 D.3 of [23], there exist $U \subset \mathbb{R}^n \times \mathbb{R}^p$, $V \coloneqq F(U) \subset \mathbb{R}^n \times \mathbb{R}^p$, and $f : V \longrightarrow U$ which is inverse of $\tilde{F}$ on U. Obviously, f has the form $f(x, y) = (\phi(x, y), y)$, where $\phi : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$. Therefore,

$$\begin{aligned} (x, y) \in U, F(x, y) = 0 &\Leftrightarrow f(0, y) = (\phi(0, y), y) = (x, y) \\ &\Leftrightarrow x = \phi(0, y). \end{aligned} \tag{40}$$

Thus, we can write $G(y) = \phi(0, y)$.

If $\mathbb{R}^p$ is replaced by any infinite dimensional Banach space Y in Theorem 24, Clarke's generalized Jacobian of the function $\tilde{F}$ above cannot be defined. In other words, we will no longer be in finite dimension to be able to apply Theorem 1 in Clarke's work [22].

This remark is very important in the rest of the work.

Remark 26. Let Y be an infinite dimensional Banach space and $F : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^n$ be a locally Lipschitz mapping on a neighborhood of a point (x_0, y_0) such that $F(x_0, y_0) = 0$. Assume that $\partial_x F(x_0, y_0)$, Clarke's generalized Jacobian is of maximal rank. Then, there exists $V \subset Y$, subset containing y_0, and a Lipschitz mapping $\varphi : V \longrightarrow U \subset \mathbb{R}^n$ such that for every $y \in V$, we have

$$F(\varphi(y), y) = 0, \varphi(y_0) = x_0. \tag{41}$$

Moreover, we have the following equivalence:

$$(x, y) \in U \times V, F(x, y) = 0 \Leftrightarrow x = \varphi(y). \tag{42}$$

Indeed, let M be a finite dimensional subspace of Y with $y_0 \in M$ and $\dim M = m (m < \infty)$. We consider the map

$$\begin{aligned} \tilde{F} : \mathbb{R}^n \times M &\longrightarrow \mathbb{R}^n, \\ (x, y) &\mapsto F(x, y). \end{aligned} \tag{43}$$

Obviously, $\tilde{F}$ is locally Lipschitz mapping, and $\partial_x \tilde{F}(x_0, y_0) = \partial_x F(x_0, y_0)$ is of maximal rank. Then, by Theorem 24, there exist $V \subset M$, open in M and containing y_0, $U \subset \mathbb{R}^n$, open containing x_0 and a locally Lipschitz mapping $\varphi : V \longrightarrow U$ such that conditions (41) and (42) hold.

Here is another result that will serve us in this work.

Theorem 27 (see [11], J. Chen). *Let* $h : [0, +\infty) \longrightarrow [0, +\infty)$ *be a continuous nondecreasing function such that*

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty. \tag{44}$$

X is a reflexive Banach space and $J : X \longrightarrow \mathbb{R}$ is a locally Lipschitz function. Assume that there exists $u_0 \in X$, $u_1 \in X$ and a bounded open neighborhood Ω of u_0 such that $u_1 \notin \Omega$ and

$$\inf_{x \in \partial\Omega} J(x) > \max \{J(u_0), J(u_1)\}. \tag{45}$$

Let $M \coloneqq \{g \in C([0, 1], X) : g(0) = u_0, g(1) = u_1\}$ and $c \coloneqq \inf_{g \in M} \max_{s \in [0,1]} J(g(s))$. If J satisfies the (h)-condition, then c is a critical value of J and $c > \max \{J(u_0), J(u_1)\}$.

Lemma 28. *Let* X *be a normed vector space and* H *be a Hilbert space equipped with the inner product* $\langle , \rangle$*. Let* $f : X \longrightarrow H$ *be a locally Lipschitz mapping. Then, the function* $\varphi : X \longrightarrow \mathbb{R}$ *defined by*

$$\varphi(x) = \langle f(x), f(x) \rangle = \| f(x) \|_H^2 \tag{46}$$

is locally Lipschitz.

Theorem 29 (see [16], Clarke). *Let* X *be a normed vector space,* $f : X \longrightarrow \mathbb{R}^n$ *be locally Lipschitz function near* $x \in X$*, and* $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a given* C^1 *function. Then,*

$$\partial(h \circ f)(x) \subset [\nabla h(f(x))] \partial f(x). \tag{47}$$

Theorem 30 (see [6], Theorem 1). *Assume that* $F : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ *is a locally Lipschitz mapping such that*

(a_1) for any $y \in \mathbb{R}^p$ the functional $\varphi_y : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$\varphi_y(x) = \frac{1}{2} \| F(x, y) \|^2 \tag{48}$$

is coercive, i.e., $\lim_{\|x\| \longrightarrow \infty} \varphi_y(x) = +\infty$

(a_2) for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^p$, the set $\partial_x F(x, y)$ is of maximal rank

Then, there exists a unique locally Lipschitz function $f : \mathbb{R}^p \longrightarrow \mathbb{R}^n$ such that equation $F(x, y) = 0$ and $x = f(y)$ are equivalent in the set $\mathbb{R}^n \times \mathbb{R}^p$.

3. Main Results

The following is a generalization of the global implicit function theorem of [6] to the case of locally Lipschitz functions from Banach spaces to Euclidean spaces.

Theorem 31. *Let Y be a real Banach space and $F : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^n$ be a locally Lipschitz function. Suppose that*

(1) for every $y \in Y$, the function φ_y defined by

$$\varphi_y(x) = \frac{1}{2} \| F(x, y) \|^2 \tag{49}$$

satisfies the (h)-condition, where $h : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous nondecreasing function such that

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty \tag{50}$$

(2) for all $(x, y) \in \mathbb{R}^n \times Y$, $\partial_x F(x, y)$ is of maximal rank

Then, there exists a unique function $f : Y \longrightarrow \mathbb{R}^n$ such that the equation "$(x, y) \in \mathbb{R}^n \times Y$ and $F(x, y) = 0$" are equivalent to $x = f(y)$. Moreover, for any finite dimensional subspace L of Y, f is locally Lipschitz on L.

Proof. Let $y \in Y$. We prove that there exists a unique element $x_y \in \mathbb{R}^n$ such that $F(x_y, y) = 0$. Indeed, φ_y is locally Lipschitz and satisfies the (h)-condition. Then, by Theorem 22, there is $x_y \in \mathbb{R}^n$ such that $\min_{\mathbb{R}^n} \varphi_y = \varphi_y(x_y)$. Since $\varphi_y = g \circ F(\cdot, y)$: $\mathbb{R}^n \longrightarrow \mathbb{R}$ and by assumption (1), the function $F(\cdot, y)$: $\mathbb{R}^n \ni x \mapsto F(x, y) \in \mathbb{R}^n$ is a locally Lipschitz mapping and $g : \mathbb{R}^n \ni x \mapsto 1/2 \|x\|^2 = 1/2 \langle x, x \rangle_{\mathbb{R}^n} \in \mathbb{R}$, it follows from Lemma 28 that φ_y is locally Lipschitz. Then, by Proposition 7, we have $0 \in \partial \phi_y(x_y)$. Moreover, according to Theorem 29, we have

$\partial \varphi_y(x_y) \subset \nabla g[F(\cdot, y) x_y] \circ \partial F(\cdot, y) x_y = \nabla g(F(x_y, y)) \circ \partial_x F(x_y, y) = \{\nabla g(F(x_y, y)) \circ x^* : x^* \in \partial_x F(x_y, y)\}$. □

Thus, there exists $x^* \in \partial_x F(x_y, y)$ such that $\nabla g(F(x_y, y)) \circ x^* = 0$, i.e.,

$$\forall v \in \mathbb{R}^n, \nabla g(F(x_y, y))[x^*(v)] = \langle F(x_y, y), x^*(v) \rangle = 0. \tag{51}$$

By assumption (2) $x^*(\mathbb{R}^n) = \mathbb{R}^n$. It follows that $F(x_y, y) = 0$.

About the uniqueness of $x_y \in \mathbb{R}^n$ such that $F(x_y, y) = 0$, we argue by contradiction supposing that there exists $x_1 \neq x_y$ in $\mathbb{R}^n$ with $F(x_1, y) = F(x_y, y) = 0$. We use Remark 26. Thus, we set $e = x_1 - x_y$, and we define the mapping $\psi_y : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\psi_y(x) \coloneqq \varphi_y(x + x_y) = \frac{1}{2} \| F(x + x_y, y) \|^2. \tag{52}$$

We have $\psi_y(0) = \psi_y(e) = 0$. Consider ψ_y on the boundary $\partial B(0, \rho)$ of the ball $B(0, \rho) \subset \mathbb{R}^n$ with some $0 < \rho < \|e\|$. By assumption (2) and Remark 26, we conclude that there exist $V \subset Y$ containing y (not necessary open in Y, but open in some finite dimensional subspace $L \subset Y$), an open subset $U \subset \mathbb{R}^n$ containing x_y, and a function $\xi : V \longrightarrow U$ such that the following equivalence holds:

$$(x, y) \in U \times V, F(x, y) = 0 \Leftrightarrow x = \xi(y). \tag{53}$$

ψ_y is also a locally Lipschitz function (so continuous), and $\partial B(0, \rho)$ is compact (by the fact that it is closed and bounded). Then, $\exists \bar{x} \in \partial B(0, \rho)$ such that

$$\psi_y(\bar{x}) = \min_{\partial B(0,\rho)} \psi_y. \tag{54}$$

We claim that there exists at least one $\rho > 0$, $\rho < \|e\|$ such that $\min_{\|x\| = \rho} \psi_y > 0$. Otherwise, we would have

$$\forall 0 < \rho < \|e\|, \min_{\|x\| = \rho} \psi_y = 0\,; \tag{55}$$

this means that for all nonnegative $\rho < \|e\|$, there exists $\bar{x} \in \mathbb{R}^n$, $\|\bar{x}\| = \rho$ such that $\psi_y(\bar{x}) = 0$. Since U is open around x_y, there exists $0 < \varepsilon < \|e\|$ such that

$$\| x - x_y \| \leq \varepsilon \Rightarrow x \in U. \tag{56}$$

Let $\bar{x} \in \mathbb{R}^n$, $\|\bar{x}\| = \varepsilon : \psi_y(\bar{x}) = 0$. Then,

$$\begin{aligned} \| (\bar{x} + x_y) - x_y \| = \|\bar{x}\| = \varepsilon, \psi_y(\bar{x}) = \frac{1}{2} \| F(\bar{x} + x_y, y) \|^2 \\ = 0 \Leftrightarrow F(\bar{x} + x_y, y) = 0. \end{aligned} \tag{57}$$

By (56) and (57), we have $(\bar{x} + x_y) \in U$, $(\bar{x} + x_y) \neq x_y$ and $F(\bar{x} + x_y, y) = 0$. It follows from (53) that $\bar{x} + x_y = \xi(y)$. Thus, $\bar{x} + x_y$ and x_y are two different elements of U with $\bar{x} + x_y =$

$\xi(y) = x_y$, what is impossible. As conclusion,

$$\exists \rho < \|e\|, \inf_{\|x\|=\rho} \psi_y > 0 = \max \left\{ \psi_y(0), \psi_y(e) \right\}. \tag{58}$$

The function ψ_y is locally Lipschitz and satisfies the (h)-condition (because φ_y satisfies this condition). Then, by (58) and Theorem 24 applied to $J = \psi_y$, we note that ψ_y has a generalized critical point v which is different from 0 and e since the corresponding critical value $\psi_y(v)$ holds

$$\psi_y(v) > \max \left\{ \psi_y(0), \psi_y(e) \right\} = 0. \tag{59}$$

We have also

$$\begin{aligned} 0 \in \partial\varphi_y(x_y + v) &\subset \nabla g\left[F(\cdot, y)(x_y + v)\right] \circ \partial F(\cdot, y)(x_y + v) \\ &= \nabla g\left(F(x_y + v, y)\right) \circ \partial_x F(x_y + v, y). \end{aligned} \tag{60}$$

This implies that $F(x_y + v, y) = 0 \Leftrightarrow \psi_y(v) = 0$. This contradiction with (59) confirms that for every $y \in Y$, there exists a unique $x_y \in \mathbb{R}^n$ such that $F(x_y, y) = 0$, and we can set $f(y) = x_y$. Of course, according to Remark 26, we can say that for any finite dimensional subspace L of Y, f is locally Lipschitz on L.

An example of function satisfying the assumptions of Theorem 31 for which Y is a Banach space is $F : \mathbb{R} \times Y \longrightarrow \mathbb{R}$ defined by

$$F(x, y) = 2x + |x| + \|y\|. \tag{61}$$

Indeed, F defined in (61) is locally Lipschitz function which is not differentiable and for any $y \in Y$ if we consider the function $\varphi_y : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\varphi_y(x) = \frac{1}{2}|F(x, y)|^2 = \frac{1}{2}(|x| + 2x + \|y\|)^2, \tag{62}$$

then φ_y is coercive and consequently satisfies the (h)-condition. Moreover, for any $(x, y) \in \mathbb{R} \times Y$, the partial generalized gradient $\partial_x F(x, y)$ defined as follows

$$\partial_x F(x, y) = \begin{cases} \{3\}, & \text{if } x > 0, \\ \{1\}, & \text{if } x < 0, \\ [1, 3], & \text{if } x = 0, \end{cases} \tag{63}$$

is of maximal rank. Namely, for any $(x, y) \in \mathbb{R} \times Y, 0 \notin \partial_x F(x, y) \subset \mathbb{R}$. Indeed, a straightforward argument shows that

$$F(x, y) = 0 \Leftrightarrow x = -\|y\|. \tag{64}$$

With the conclusion about the regularity of f in Theorem 31, we cannot expect in general the continuity of f on the whole Y. Here is a counterexample.

Remark 32. Let us set $Y = \ell_1$, where ℓ_1 stands for the space of real sequences $(u_m)_{m\in\mathbb{N}}$ such that $\sum_{m=0}^{\infty}|u_m| < \infty$, endowed with the nonequivalent norms:

$$\begin{aligned} \|u_m\|_1 &= \sum_{m=0}^{\infty}|u_m|, \\ \|u_m\|_2 &= \left(\sum_{m=0}^{\infty}|u_m|^2\right)^{1/2}. \end{aligned} \tag{65}$$

Indeed, for $m \in \mathbb{N}^*$, we define $X^m := (1, 1/2, 1/3, \cdots, 1/m, 0, \cdots) \in \ell_1$. We claim that $(X^m)_m$ is a bounded sequence with respect to the norm $\|\cdot\|_2$ which is unbounded with respect to $\|\cdot\|_1$. For $m \in \mathbb{N}^*$, we have

$$\begin{aligned} \|X^m\|_2 &= \left(\sum_{k=1}^{m}\frac{1}{k^2}\right)^{1/2}, \\ \|X^m\|_1 &= \sum_{k=1}^{m}\frac{1}{k}. \end{aligned} \tag{66}$$

Then,

$$\lim_{m\longrightarrow+\infty}\|X^m\|_2 = \left(\sum_{k=1}^{+\infty}\frac{1}{k^2}\right)^{1/2} < +\infty, \tag{67}$$

$$\lim_{m\longrightarrow+\infty}\|X^m\|_1 = \sum_{k=1}^{+\infty}\frac{1}{k} = +\infty. \tag{68}$$

Now, let us consider the canonical injection $\mathcal{J} : (\ell_1, \|\cdot\|_2) \longrightarrow (\ell_1, \|\cdot\|_1)$. It is obvious by (67) that $\mathcal{J}$ is not continuous on ℓ_1. However, for any finite dimensional subspace L of ℓ_1, since the restriction of these norms on L are equivalent on L, it follows that $\mathcal{J}_L : (L_1, \|\cdot\|_2) \longrightarrow (L_1, \|\cdot\|_1)$ is Lipschitz.

We add some technical hypothesis to those of Theorem 31 in order to obtain the continuity of the implicit function f.

Definition 33. Let X, Y be two normed vector spaces. We say that a function $F : X \times Y \longrightarrow \mathbb{R}$ is coercive with respect to x (the first variable), locally uniformly with respect to y (the second variable), if for any $\bar{y} \in Y$, there exists an open neighborhood V of $\bar{y}$ in Y such that

$$\lim_{\|x\|\longrightarrow\infty} \inf_{y\in V} F(x, y) = +\infty. \tag{69}$$

Lemma 34. *Let E be a Euclidean space. Then, every bounded sequence with a unique limit point is convergent.*

Proof. Let $(x_m)_{m\geq0}$ be a sequence of E which has a unique limit point $\bar{x} \in E$. This implies that any subsequence of $(x_m)_{m\geq0}$ has a subsequence converging to $\bar{x}$ by the Bolzano Weierstrass theorem. We argue by contradiction assuming that $(x_m)_{m\geq0}$ is not convergent. Then, there exists $\varepsilon > 0$ such that

$$\text{for any } k > 0, \text{ there exists } m_k \geq k, \text{ with } \|x_{m_k} - \bar{x}\| > \varepsilon. \quad (70)$$

$(x_{m_k})_k$ must have a subsequence $(x_{m_{k_i}})_i$ such that

$$x_{m_{k_i}} \longrightarrow \bar{x}, \quad (71)$$

which contradicts (70). □

Theorem 35. *Let Y be a real Banach space and $F : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^n$ be locally Lipschitz. Suppose that*

(1) the function $\chi : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}$ defined by

$$\chi(x, y) = \frac{1}{2}\|F(x, y)\|^2 \quad (72)$$

is coercive with respect to x, locally uniformly with respect to y

(2) for all $(x, y) \in \mathbb{R}^n \times Y$, $\partial_x F(x, y)$ is of maximal rank

Then, there exists a unique function $f : Y \longrightarrow \mathbb{R}^n$ such that

$$\begin{gathered} (x, y) \in \mathbb{R}^n \times Y, \\ F(x, y) = 0 \Leftrightarrow x = f(y). \end{gathered} \quad (73)$$

Moreover, f is continuous on the whole Y.

Proof. Let $y \in Y$. We consider the function $\varphi_y : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$\varphi_y(x) := \frac{1}{2}\|F(x, y)\|^2. \quad (74)$$

□

Since φ_y is coercive (because χ is coercive with respect to x, locally uniformly with respect to y), it follows that for any continuous nondecreasing function $h : [0,+\infty) \longrightarrow [0,+\infty)$ such that

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty, \quad (75)$$

φ_y satisfies the (h)-condition. Moreover, F is locally Lipschitz. So, by Theorem 31, we conclude that there exists a unique global implicit function $f : Y \longrightarrow \mathbb{R}^n$ such that

$$\begin{gathered} (x, y) \in \mathbb{R}^n \times Y, \\ F(x, y) = 0 \Leftrightarrow x = f(y). \end{gathered} \quad (76)$$

It remains to show that f is continuous on the whole Y. For this, let $(y_m)_{m\in\mathbb{N}} \subset Y$ be sequence such that

$$y_m \longrightarrow \bar{y} \in Y. \quad (77)$$

For all $m \in \mathbb{N}$, $F(f(y_m), y_m) = 0$. This implies that the sequence $(\chi(x_m, y_m))_m = (\|F(f(y_m), y_m)\|)_m$ is bounded. Since χ is coercive with respect to x, locally uniformly with respect to y, there exists an open subset $\mathcal{Q} \subset Y$ containing $\bar{y}$ such that

$$\lim_{\|x\|\longrightarrow\infty} \inf_{y\in\mathcal{Q}} \chi(x, y) = \lim_{\|x\|\longrightarrow\infty} \inf_{y\in\mathcal{Q}} \frac{1}{2}\|F(x, y)\|^2 = +\infty. \quad (78)$$

In addition, by the convergence of y_m to $\bar{y}$, there exists $m_0 \in \mathbb{N}$ such that

$$m > m_0 \Rightarrow y_m \in \mathcal{Q}. \quad (79)$$

So,

$$\text{for } m > m_0, \ \frac{1}{2}\|F(x_m, y_m)\|^2 \geq \inf_{y\in\mathcal{Q}} \frac{1}{2}\|F(x_m, y)\|^2. \quad (80)$$

According to (78) and (80), we conclude that the sequence $(x_m)_{m\in\mathbb{N}} := (f(y_m))$ is bounded in $\mathbb{R}^n$. Let $\bar{x}$ be a limit point of $(x_m)_m$. Thus, there exists a convergent subsequence $(x_{m_k})_k$ of (x_m) such that

$$x_{m_k} \longrightarrow \bar{x} \in \mathbb{R}^n. \quad (81)$$

On the other hand, for all $k \in \mathbb{N}$, $F(x_{m_k}, y_{m_k}) = 0$. Then, it follows from (77), (81), and the continuity of the function F that

$$0 = \lim_{k\longrightarrow+\infty} F\left(x_{m_k}, y_{m_k}\right) = F(\bar{x}, \bar{y}). \quad (82)$$

Thus, we have

$$F(\bar{x}, \bar{y}) = 0 \Leftrightarrow \bar{x} = f(\bar{y}). \quad (83)$$

So, $(x_m)_m$ has a unique limit point $\bar{x} = f(\bar{y})$. So, by Lemma 34,

$$x_m \longrightarrow \bar{x}, \quad (84)$$

that is,

$$f(y_m) \longrightarrow f(\bar{y}). \quad (85)$$

From (77) and (85), f is continuous on Y.

As a consequence of our Theorem 31, we have the following nonsmooth global inverse function theorem.

Theorem 36. *Assume $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a locally Lipschitz mapping such that*

(1) for any $y \in \mathbb{R}^n$, there exists a continuous nondecreasing function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\int_0^\infty \frac{ds}{1+h(s)} = +\infty, \tag{86}$$

and the functional $\varphi_y : \mathbb{R}^n \longrightarrow \mathbb{R}$ *defined by*

$$\varphi_y(x) = \frac{1}{2}\|f(x) - y\|^2 \tag{87}$$

satisfies the (h)*-condition*

(2) *for any* $x \in \mathbb{R}^n$, *we have that* $\partial f(x)$ *is of maximal rank*

Then, f *is a global homeomorphism on* $\mathbb{R}^n$ *and* f^{-1} *is locally Lipschitz.*

Corollary 37 (see [25], Hadamard-Palais). *Let* X, Y *be finite dimensional Banach spaces. Assume that* $f : X \longrightarrow Y$ *is a* C^1 *-mapping such that*

(1) $\lim_{\|x\| \longrightarrow \infty} \|f(x)\| = \infty$

(2) *for any* $x \in X$, $f'(x)$ *is invertible*

then f *is a diffeomorphism.*

Question. Is it still possible the conclusion of Theorem 31 under the assumption of the (h)-condition on the function $\tau_y : x \mapsto \|F(x, y)\|^\alpha$, where α is a positive constant different from 2?

In fact, according to our two Lemmas 42 and 43 and Corollary 44, it is enough to assume that τ_y is locally Lipschitz in the case $0 < \alpha < 2$. Else, this additional hypothesis is not need in the case $\alpha > 2$.

Therefore, we have the following result from Theorem 31.

Theorem 38. *Let* Y *be a real Banach space and* $F : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^n$ *be a locally Lipschitz mapping. Suppose that*

(1) *for any* $y \in Y$, *there exists* $0 < \alpha < 2$, *so that the function* $\tau_y : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\tau_y(x) = \|F(x, y)\|^\alpha \tag{88}$$

is locally Lipschitz and satisfies the (h)*-condition, where* $h : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous nondecreasing function such that*

$$\int_0^\infty \frac{ds}{1+h(s)} = +\infty \tag{89}$$

(2) *for all* $(x, y) \in \mathbb{R}^n \times Y$, $\partial_x F(x, y)$ *is of maximal rank*

Then, there exists a unique function $f : Y \longrightarrow \mathbb{R}^n$ *such that the equation* ˝$(x, y) \in \mathbb{R}^n \times Y$ *and* $F(x, y) = 0$˝ *are equivalent to* $x = f(y)$. *Moreover, for any finite dimensional subspace* L *of* Y, f *is locally Lipschitz on* L.

Proof. Let $y \in Y$. We notice that $\tau_y = [2\varphi_y]^{\alpha/2}$, where φ_y is defined by (49). Since τ_y satisfies the (h)-condition, it follows from Lemma 43 and Corollary 44 that φ_y satisfies the (h)-condition. Thus, we achieve the proof by using Theorem 31. □

But, what happens if we replace $\mathbb{R}^n$ by any Banach space X in the domain of the function F?

Theorem 39. *Let* X, Y *be Banach spaces and* Z *be Euclidean space such that* $\dim Z = n < \infty$. *Let* $F : X \times Y \longrightarrow Z$ *be locally Lipschitz function. Assume that*

(1) *for all* $y \in Y$, *the function* $\varphi_y : X \longrightarrow \mathbb{R}$ *defined by*

$$\phi_y(x) = \frac{1}{2}\|F(x, y)\|^2 \tag{90}$$

satisfies the (h)*-condition, where* $h : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous nondecreasing function such that*

$$\int_0^\infty \frac{ds}{1+h(s)} = +\infty \tag{91}$$

(2) *for any finite dimensional subspace* L *of* X *with* $\dim L = n$ *and for all* $(x, y) \in L \times Y$, $\partial_x F_L(x, y)$ *is of maximal rank*

Then,

$$\begin{gathered}(x, y) \in X \times Y,\\ F(x, y) = 0 \Leftrightarrow x = 0.\end{gathered} \tag{92}$$

Proof. We use Theorem 31 in order to prove this result. Firstly, we prove that there exists a unique global implicit function $f : Y \longrightarrow X$ such that $(x, y) \in X \times Y$ and $F(x, y) = 0$ are equivalent from $x = f(y)$. After that, we will claim that $f \equiv 0$ on Y.

Let $y \in Y$. Since φ_y is bounded form below, locally Lipschitz, and satisfies the (h)-condition, we see by Theorem 22 that φ_y has a minimum which is achieved at a critical point $x_y \in X$. Let L be a finite dimensional subspace of X such that $x_y \in L$ and $\dim L = n$. □

Consider functions $\tilde{F} : L \times Y \longrightarrow Z$ and $\tilde{\varphi}_y : L \longrightarrow \mathbb{R}$ defined, respectively, by

$$\tilde{F}(x, y) = F(x, y), \tilde{\varphi}_y(x) = \varphi_y(x) = \frac{1}{2}\|F(x, y)\|^2 = \frac{1}{2}\left\|\tilde{F}(x, y)\right\|^2. \tag{93}$$

By assumption (1), the function $\tilde{F}$ is locally Lipschitz and $\tilde{\varphi}_y$ is then locally Lipschitz as composition of $\tilde{F}$ by C^1 function g where

$$g : Z \longrightarrow \mathbb{R}\,;\, x \mapsto \frac{1}{2}\|x\|^2 = \frac{1}{2}\langle x, x\rangle_Z. \tag{94}$$

Likewise, $\tilde{\varphi}_y$ satisfies the (h)-condition. It follows that $\tilde{\varphi}_y$ has a minimum on L. Since $x_y \in L$, it is obvious that $\min_{x\in L}\tilde{\varphi}_y(x) = \varphi(x_y)$. Thus, by Theorem 29, we obtain

$$\begin{aligned} 0 \in \partial\tilde{\varphi}_y(x_y) = \partial_L\varphi_y(x_y) &\subset \nabla g(\tilde{F}(x_y, y)) \circ \partial_x\tilde{F}(x_y, y) \\ &= \nabla g(F(x_y, y)) \circ \partial_x F_L(x_y, y). \end{aligned} \tag{95}$$

This means that there exists $x^* \in \partial_x F_L(x_y, y)$ such that

$$\forall h \in L, \langle F(x_y, y), x^*(h)\rangle = 0. \tag{96}$$

Thus, we conclude by assumption (2) that $\tilde{F}(x_y, y) = F(x_y, y) = 0$. It is clear by Theorem 31 that x_y is the only solution of the equation $F(x_y, y) = 0$ in L. But the question is the uniqueness of this solution in all X. Even though it is unpredictable, we will answer yes to this question in the following. About uniqueness of $x_y \in X$ such that $F(x_y, y) = 0$, we argue by contradiction.

Suppose that there exists $x_1 \neq x_y$ such that $F(x_1, y) = F(x_y, y) = 0$. We choose then another finite dimensional subspace of X (that we note again L here to keep the same notation) which contains both x_1 and x_y, such that dim $L = n$. We consider the same functions defined in (93), but with the subspace L that we choose here. We find that $\min_{x\in L}\tilde{\varphi}_y(x) = \tilde{\varphi}_y(x_y)$ and by [15] (Proposition 6), Theorem 29, and assumption (2), we conclude that $\tilde{F}(x_y, y) = 0$. Considering the function $\tilde{\psi}_y$ defined as in (52) by

$$\tilde{\psi}_y(x) \coloneqq \tilde{\varphi}_y(x + x_y) = \frac{1}{2}\|F(x + x_y, y)\|^2 \tag{97}$$

and following the same approach in the proof of Theorem 31, we come to the contradiction. Thus, there exists a unique global implicit function $f : Y \longrightarrow X$ such that $F(f(y), y) = 0$ for all $y \in Y$.

It remains to be shown that $f \equiv 0$ on Y. Indeed, since X is infinite dimensional Banach space, it is possible to find two n-dimensional subspaces L_1 and L_2 of X such that $L_1 \cap L_2 = \{0_X\}$. Let $F_1 : L_1 \times Y \longrightarrow Z$ and $F_2 : L_2 \times Y \longrightarrow Z$ be the function defined by

$$\begin{aligned} F_1(x, y) &= F(x, y), \\ F_2(x, y) &= F(x, y). \end{aligned} \tag{98}$$

By assumptions (1) and (2), both functions F_1 and F_2 verify the assumptions of Theorem 31. Consequently, there exist two functions $\varphi_1 : Y \longrightarrow L_1$ and $\phi_2 : Y \longrightarrow L_2$ such that $\forall y \in Y, F(\varphi_1(y), y) = F(\varphi_2(y), y) = 0$. Then, according to the uniqueness of $x_y \in X$ such that $F(x_y, y) = 0$, we have $\varphi_1(y) = f(y) = \varphi_2(y) \in L_1 \cap L_1$. Thus, $f(y) = 0, \forall y \in Y$.

In virtue of Lemma 43, the following theorem is a consequence of Theorem 39.

Theorem 40. *Let X, Y be Banach spaces and Z be Euclidean space such that $\dim Z = n < \infty$. Let $F : X \times Y \longrightarrow Z$ be locally Lipschitz function. Assume that*

(1) for every $y \in Y$, there exists $0 < \alpha < 2$ such that the function $\tau_y : X \longrightarrow \mathbb{R}$ defined by

$$\tau_y(x) = \|F(x, y)\|^\alpha \tag{99}$$

is locally Lipschitz and satisfies the (h)-condition, where $h : [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous nondecreasing function such that

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty \tag{100}$$

(2) for any finite dimensional subspace L of X and for any $(x, y) \in L \times Y$, $\partial_x F_L(x, y)$ is of maximal rank

Then.

$$\begin{gathered} (x, y) \in X \times Y, \\ F(x, y) = 0 \Leftrightarrow x = 0. \end{gathered} \tag{101}$$

Remark 41. If $\alpha > 2$, it is useless to add the locally Lipschitz condition to the (h)-condition for the first assumption of Theorems 38 and 40. Indeed, in Theorem 38, for example, since φ_y is locally Lipschitz and $\alpha/2 > 1$, it follows from Lemma 43 that $\tau_y = [2\varphi_y]^{\alpha/2}$ is also locally Lipschitz. Moreover, φ_y satisfies the (h)-condition.

Lemma 42. *Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ be a locally Lipschitz function. Let $\alpha > 1$. If g^α is locally Lipschitz, then for any $x, v \in \mathbb{R}^n$ with $v \neq 0$, we have*

$$[g^\alpha]^0(x\,; v) = \alpha \cdot [g(x)]^{\alpha-1} g^0(x\,; v). \tag{102}$$

Proof. Let $(w_m)_m$ be a sequence in $\mathbb{R}^n$ and $(t_m)_m \subset (0;+\infty)$

another sequence such that

$$\begin{aligned} w_m &\longrightarrow x, \\ t_m &\longrightarrow 0^+. \end{aligned} \quad (103)$$

□

For fixed m, the function $\mu : I_m \longrightarrow \mathbb{R}\,;\, t \mapsto t^\alpha$ is differentiable, where

$$I_m := \{\theta g(a_m) + (1-\theta)g(b_m)\,;\,\theta \in [0,1]\}, \text{ with } a_m = w_m + t_m v,\ b_m = w_m. \quad (104)$$

Now, it is known that there exists $c_m \in I_m$ such that

$$\begin{aligned} [g(a_m)]^\alpha - [g(b_m)]^\alpha &= \mu[g(a_m)] - \mu[g(b_m)] \\ &= \mu'(c_m)[g(a_m) - g(b_m)] \\ &= \mu'[\theta_m g(a_m) + (1-\theta_m)g(b_m)] \cdot [g(a_m) - g(b_m)] \\ &= \alpha \cdot [\theta_m g(a_m) + (1-\theta_m)g(b_m)]^{\alpha-1} \cdot [g(a_m) - g(b_m)] \\ &= \mathscr{K}_m \cdot [g(w_m + t_m v) - g(w_m)], \end{aligned} \quad (105)$$

where

$$\mathscr{K}_m = \alpha \cdot [\theta_m g(w_m + t_m v) + (1-\theta_m)g(w_m)]^{\alpha-1}, \quad (106)$$

with $\theta_m \in [0,1]$. Then, we have

$$\frac{[g]^\alpha(w_m + t_m v) - [g]^\alpha(w_m)}{t_m} = \mathscr{K}_m \cdot \frac{g(w_m + t_m v) - g(w_m)}{t_m}. \quad (107)$$

Since g is continuous, there exists a neighborhood $\mathscr{V}$ of x and $\mathscr{K} > 0$ such that

$$|g(z)| \le \mathscr{K}, \forall z \in \mathscr{V}. \quad (108)$$

It follows from the convergence of (w_m, t_m) to $(x, 0)$ and the continuity of g and (108) that

$$\lim_{m \longrightarrow +\infty} \mathscr{K}_m = \alpha \cdot [g(x)]^{\alpha-1}. \quad (109)$$

By (107), (109), and the fact that

$$\limsup_{\substack{w \longrightarrow x \\ t \longrightarrow 0^+}} \frac{g(w + tv) - g(w)}{t} = g^0(x\,;\,v), \quad (110)$$

we conclude that

$$[g^\alpha]^0(x\,;\,v) = \alpha \cdot [g(x)]^{\alpha-1} g^0(x\,;\,v). \quad (111)$$

Lemma 43. *Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ be a locally Lipschitz function. Let $\alpha > 1$ and $h : [0,+\infty) \longrightarrow [0,+\infty)$ a continuous nondecreasing function such that*

$$\int_0^\infty \frac{ds}{1 + h(s)} = +\infty. \quad (112)$$

Then, $g^\alpha(x) := [g(x)]^\alpha$ is locally Lipschitz function. Moreover, g satisfies the (h)-condition if and only if g^α satisfies the (h)-condition.

Proof. Let $\bar{x} \in X$. There exist $V \ni \bar{x}$ open subset of $\mathbb{R}^n$ and $k > 0$ such that

$$|g(x) - g(y)| \le k\|x - y\|, \forall x, y \in V. \quad (113)$$

□

Let $\rho > 0$ such that $\bar{B}_\rho(\bar{x}) := \{x \in \mathbb{R}^n : \|\bar{x} - x\| \le \rho\} \subset V$. For $x, y \in \bar{B}_\rho$, as in the previous Lemma 42, there exists $\theta \in [0,1]$ such that we have

$$|g^\alpha(x) - g^\alpha(y)| = \alpha \cdot |g(x) - g(y)| \cdot [\theta g(x) + (1-\theta)g(y)]^{\alpha-1}. \quad (114)$$

g is continuous, and $\bar{B}_\rho$ is compact. Let

$$M = \max_{z \in \bar{B}_\rho(\bar{x})} |g(z)|. \quad (115)$$

Then, we have

$$|\theta g(x) + (1-\theta)g(y)|^{\alpha-1} \le M^{\alpha-1}. \quad (116)$$

It follows from (113), (114), and (116) that we have

$$|g^\alpha(x) - g^\alpha(y)| \le kM^{\alpha-1}\|x - y\|, \forall x, y \in B_\rho(\bar{x}), \quad (117)$$

where $B_\rho(\bar{x}) := \{x \in \mathbb{R}^n : \|\bar{x} - x\| < \rho\} \subset V$ is open. Then, g^α is locally Lipschitz.

For the second part of Lemma 43, just specify that $(u_m)_{m\ge0}$ is a (h)-sequence of g if and only if $(u_m)_{m\ge0}$ is a (h)-sequence of g^α.

Let $(v_m)_{m\ge0} \subset \mathbb{R}^n$ be a (h)-sequence of g. Then, there exist $q > 0$, $(\tau_m)_m \subset (0,+\infty)$ with $\tau_m \longrightarrow 0^+$, such that

$$g(v_m) \le q, \forall m \ge 0, \quad (118)$$

$$g^0(v_m\,;\,v - v_m)(1 + h(\|v_m\|)) \ge -\tau_m\|v - v\|_m, \forall v \in \mathbb{R}^n. \quad (119)$$

It follows from (118) that $[g(v_m)]^\alpha$ is bounded. From Lemma 42 and inequality (119), we deduce that

$$\begin{aligned} &[g^\alpha]^0(v_m\,;\,v - v_m)(1 + h(\|v_m\|)) \\ &\quad \ge -\bar{\tau}_m\|v - v_m\|, \forall v \in \mathbb{R}^n, \bar{\tau}_m := \tau_m \alpha q^{\alpha-1} \longrightarrow 0^+. \end{aligned} \quad (120)$$

Thus, $(v_m)_{m\ge0}$ is a (h)-sequence of g^α.

Conversely, let $(u_m)_{m\geq 0} \subset \mathbb{R}^n$ be a (h)-sequence of g^α. Then, there exists $p>0$ such that

$$[g(u_m)]^\alpha \leq p, \forall m \geq 0, \tag{121}$$

$$[g^\alpha]^0(u_m; v-u_m)(1+h(\|u_m\|)) \geq -\varepsilon_m\|v-u_m\|, \forall v \in \mathbb{R}^n, \varepsilon_m \longrightarrow 0^+. \tag{122}$$

It follows from (121) that there exists $\bar{p}>0$ such that

$$g(u_m) \leq \bar{p}, \forall m \geq 0. \tag{123}$$

By Lemma 42, (122), and (123), we have

$$g^0(u_m; v-u_m)(1+h(\|u_m\|)) \geq -\delta_m\|v-u_m\|, \forall v \in \mathbb{R}^n, \delta_m := \frac{\varepsilon_m}{\alpha \bar{p}^{(\alpha-1)}} \longrightarrow 0^+. \tag{124}$$

Therefore, $(u_m)_{m\geq 0}$ is also a (h)-sequence of g. Then, for any $\alpha>1$, g satisfies the (h)-condition if and only if g^α satisfies the (h)-condition.

But what about $0<\alpha<1$?

Corollary 44. *Let $g:\mathbb{R}^n \longrightarrow \mathbb{R}_+$ be a function and $0<\alpha<1$ such that g^α is locally Lipschitz. Then, g is locally Lipschitz and for any continuous nondecreasing function $h:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that*

$$\int_0^\infty \frac{ds}{1+h(s)} = +\infty. \tag{125}$$

g satisfies the (h)-condition if and only if g^α satisfies the (h)-condition.

Proof. We notice that $g=(g^\alpha)^{1/\alpha}$ and $1/\alpha>1$. Then, we apply Lemma 43. □

4. Example of Noncoercive Function Satisfying the (h)-Condition

To illustrate that the compactness condition allowing to obtain the existence of a global implicit function in our main results is weaker than that used in Theorem 30, we provide in this section an example of a noncoercive and locally Lipschitz function satisfying the (h)-condition. We follow the idea used by Chen and Tang in [13] (Theorem 3.3).

Let $1<p<\infty$. Define

$$L^p(0,T;\mathbb{R}^N) = \left\{u \in L^1(0,T;\mathbb{R}^N): \int_0^T |u(t)|^p dt < \infty\right\}, \tag{126}$$

with the norm

$$\|u\|_p = \left(\int_0^T |u|^p dt\right)^{1/p}. \tag{127}$$

For $u \in L^1_{loc}(0,T;\mathbb{R}^N)$, u' is said to be the weak derivative of u, if $u' \in L^1_{\text{loc}}(0,T;\mathbb{R}^N)$ and

$$\int_0^T u'\phi dt = -\int_0^T u\phi' dt, \forall \phi \in C_0^\infty(0,T;\mathbb{R}^N). \tag{128}$$

Let

$$W_0^{1,p}(0,T;\mathbb{R}^N) = \left\{u \in L^p(0,T;\mathbb{R}^N): u(0)=u(T), u' \in L^p(0,T;\mathbb{R}^N)\right\}. \tag{129}$$

$W_0^{1,p}(0,T;\mathbb{R}^N)$ is a reflexive Banach space (see [13]) with the norm

$$\|u\|_{W_0^{1,p}(0,T;\mathbb{R}^N)} = \left[\int_0^T \left(|u|^p + |u'|^p\right) dt\right]^{1/p}. \tag{130}$$

Remark 45 (see [13]). We have the following direct decomposition of $W_0^{1,p}(0,T;\mathbb{R}^N)$

$$W_0^{1,p}(0,T;\mathbb{R}^N) = \mathbb{R}^N \oplus V, \text{where} V = \left\{v \in W_0^{1,p}(0,T;\mathbb{R}^N): \int_0^T v(t)dt = 0\right\}. \tag{131}$$

Consider now the following functional:

$$J(u) = \int_0^T \frac{1}{p}|u'|^p dt, u \in W_0^{1,p}(0,T;\mathbb{R}^N). \tag{132}$$

We know that (see [26]) $J \in C^1(W_0^{1,p}(0,T;\mathbb{R}^N),\mathbb{R})$ and p-Laplacian operator $u \mapsto (|u'|^{p-2}u')'$ is the derivative operator of J in the weak sense. That is,

$$A = J' : W_0^{1,p}(0,T;\mathbb{R}^N) \longrightarrow \left(W_0^{1,p}(0,T;\mathbb{R}^N)\right)^*, \tag{133}$$

$$\langle A(u), v\rangle = \int_0^T \left(|u'(t)|^{p-2}u'(t), v'(t)\right)_{\mathbb{R}^N} dt, u, v \in W_0^{1,p}(0,T;\mathbb{R}^N). \tag{134}$$

Proposition 46 (see [27], Fan and Zhao). *J' is a mapping of $(S)_+$, i.e., if*

$$u_m \rightharpoonup u, \quad \limsup_{m\longrightarrow\infty}\left(J'(u_m) - J'(u), u_m - u\right) \leq 0, \tag{135}$$

then $(u_m)_m$ has a convergent subsequence in $W_0^{1,p}(0,T;\mathbb{R}^N)$.

For every $u \in W_0^{1,p}(0,T;\mathbb{R}^N)$, set

$$\bar{u} = \frac{1}{T}\int_0^T u(t)dt, \tag{136}$$

$$\tilde{u}(t) = u(t) - \bar{u}. \tag{137}$$

We have the following Poincare-Wirtinger inequality (see [28]):

$$\exists a > 0 \text{ such that } \|\tilde{u}\|_\infty \le a\|u'\|_p, \forall u \in W_0^{1,p}(0, T; \mathbb{R}^N). \tag{138}$$

We consider the functional $\phi : W_0^{1,p}(0, T; \mathbb{R}^N) \longrightarrow \mathbb{R}$ defined by

$$\phi(u) = \int_0^T \frac{1}{p}|u'|^p dt - \int_0^T j(t, u)dt, u \in W_0^{1,p}(0, T; \mathbb{R}^N), \tag{139}$$

where $j(t, u)$: $[0, T] \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is the norm $|\cdot|$ on $\mathbb{R}^N$ defined by

$$j(t, u) = |u| = \left(\sum_{i=1}^N |u_i|^2\right)^{1/2}, \text{ for } u := (u_1, u_2, \cdots, u_N) \in \mathbb{R}^N. \tag{140}$$

Let $h(s) = s$. Following the same approach as done by Chen and Tang in [13], we will show that the function φ satisfies the (h)-condition and is noncoercive on all of $W_0^{1,p}(0, T; \mathbb{R}^N)$.

We show that the function j satisfies the following assumptions:

(1) For all $u \in \mathbb{R}^N, t \mapsto j(t, u)$ is measurable

(2) For almost all $t \in [0, T], u \mapsto j(t, u)$ is locally Lipschitz

(3) For every $r > 0$, there exists $\alpha_r \in L^1([0, T])$ such that for almost all $t \in [0, T], |u| \le r$ and all $w \in \partial j(t, u)$, we have $|w| \le \alpha_r(t)$, where $\partial j(t, s)$ is Clarke's generalized gradient of j with respect to the variable s

(4) There exist $0 < \mu < p$ and $M > 0$ such that for almost all $t \in [0, T]$ and all $|u| \ge M$, we have

$$j^0(t, u; u) < \mu j(t, u) \tag{141}$$

(5) $j(t, u) \longrightarrow +\infty$ uniformly for almost all $t \in [0, T]$ as $|u| \longrightarrow \infty$

Obviously, the function j defined in (140) satisfies conditions (1), (2), and (5). In addition, for $(t, u) \in [0, T] \times \mathbb{R}^N$, we have the following:

(1) If $u \neq 0, \partial j(t, u) = \{u/|u|\}$

(2) If $u = 0, \partial j(t, 0) = \{w \in \mathbb{R}^N : |y| \ge \langle w, y\rangle$, for any $y \in \mathbb{R}^N\} = \bar{B}(0, 1)$

That is,

$$\partial j(t, 0) = \bar{B}(0, 1) := \{y \in \mathbb{R}^N : |y| \le 1\}. \tag{142}$$

Indeed, for $t \in [0, T]$, the function $j(t, \cdot)$ is convex. Then, Clarke's generalized gradient $\partial j(t, u)$ of $j(t, \cdot)$ at a point u coincides with the subdifferential of $j(t, \cdot)$ in convex analysis sense (see Proposition 5). We recall also that the norm in Hibert space is Fréchet differentiable at any point $u \neq 0$.

Thus,

$$|w| \le 1, \text{for all } w \in \partial j(t, u). \tag{143}$$

Consequently, according to (143), for every $r > 0$, and $\alpha_r(t) = 1$, then $\alpha_r \in L^1([0, T])$ and

$$|u| \le r \Rightarrow |w| \le \alpha_r, \forall w \in \partial j(t, u). \tag{144}$$

Thus, the function j satisfies the assumption (3).

On other hand, since $1 < p$, taking $\mu = 1 + p/2$, we have

$$0 < 1 < \mu < p. \tag{145}$$

Moreover, for $M > 0$, we have

$$|u| \ge M \Rightarrow u \neq 0, \tag{146}$$

$$\mu j(t, u) = \mu|u|. \tag{147}$$

Then,

$$j^0(t, u; u) = \left\langle \frac{u}{|u|}, u \right\rangle = |u|. \tag{148}$$

It follows from (145), (146), and (148) that

$$|u| \ge M \Rightarrow j^0(t, u; u) < \mu j(t, u). \tag{149}$$

Thus, the function j satisfies the assumption (4).

Under the previous assumptions, φ is locally Lipschitz (see [13] (Theorem 3.3)).

(1) By (130), for any $u \in \mathbb{R}^N$, we have

$$\begin{gathered} \|u\| = T^{1/p}|u|, \int_0^T \frac{1}{p}|u'|^p dt = 0, \\ \phi(u) = -\int_0^T |u| dt = -T|u|. \end{gathered} \tag{150}$$

Then,

$$\lim_{\substack{\|u\| \longrightarrow +\infty \\ u \in \mathbb{R}^N}} \phi(u) = \lim_{|u| \longrightarrow +\infty} -T|u| = -\infty. \tag{151}$$

Thus, φ is not coercive.

(2) Let $\{u_m\}_{m\geq 1}$ be a (h)-sequence of φ, i.e., there exists $M_1>0$ such that

$$|\varphi(u_m)|\leq M_1, \tag{152}$$

$$\lim_{m\longrightarrow+\infty}(1+\|u_m\|)\gamma(u_m)=0, \tag{153}$$

where

$$\gamma(u_m)=\min_{w^*\in\partial\phi(u_m)}\|w^*\|. \tag{154}$$

Without loss of generality, we suppose that $u_m\neq 0, \forall m\geq 1$.

According to Proposition 6, let $u_m^*\in\partial\phi(u_m)$ such that $\|u_m^*\|=\gamma(u_m)$.

By definition (134) of operator A, we have

$$u_m^*=A(u_m)-w_m, \tag{155}$$

with $w_m\in\partial j(t,u_m)$.

From the second assertion of (152), we have

$$\langle u_m^*,u_m\rangle=\int_0^T|u_{m'}(t)|^p dt-\int_0^T(w_m(t),u_m(t))dt\leq\varepsilon_m,\varepsilon_m\downarrow 0. \tag{156}$$

Thus, it follows from Definition 4 and inequality (156) that

$$\int_0^T|u_{m'}(t)|^p dt-\int_0^T j^0(t,u_m(t);u_m(t))dt\leq\varepsilon_m. \tag{157}$$

Since $u_m\neq 0$, according to (148), the inequality (157) implies

$$\int_0^T|u_{m'}(t)|^p dt-\int_0^T|u_m(t)|dt\leq\varepsilon_m. \tag{158}$$

From the first assertion of (152), we have

$$-\frac{\mu}{p}\int_0^T\left|u_m'(t)\right|^p dt+\int_0^T\mu|u_m(t)|dt\leq\mu M_1. \tag{159}$$

It follows from (158) and (159) that

$$\left(1-\frac{\mu}{p}\right)\int_0^T\left|u_m'(t)\right|^p dt+\int_0^T(\mu-1)|u_m(t)|dt\leq M_m, \tag{160}$$

with $M_m=\varepsilon_m+\mu M_1$. By (160), we have

$$\left(1-\frac{\mu}{p}\right)\int_0^T\left|u_m'(t)\right|^p\leq M_m, m\geq 1, M_m\longrightarrow\mu M_1. \tag{161}$$

By (161), there exists $M_0>0$ such that for $t\in[0,T]$ and $m\geq 1$,

$$\left|u_m'(t)\right|\leq M_0. \tag{162}$$

From (161) and the Poincare-Wirtinger inequality (138), $\{\tilde{u}_m\}$ is bounded in $W_0^{1,p}(0,T;\mathbb{R}^N)$. By exploiting (152) once again, we use (136) to have

$$\left|\frac{1}{p}\int_0^T|\tilde{u}_{m'}|^p dt-\int_0^T|u_m(t)|dt\right|\leq M_1, m\geq 1. \tag{163}$$

Since $\{\tilde{u}_m\}$ is bounded, it follows from (163) that there exists $M_2>0$ such that

$$\int_0^T|u_m(t)|dt\leq M_2,\forall m\geq 1. \tag{164}$$

Thus, there exists $M_3>0$ such that for $t\in[0,T]$ and $m\geq 1$,

$$|u_m(t)|\leq M_3. \tag{165}$$

By (162) and (165), we infer that $\{u_m\}_{m\geq 1}\subset W_0^{1,p}(0,T;\mathbb{R}^N)$ is bounded, and so by passing to a subsequence if necessary, we may assume that

$$\begin{aligned}&u_m\rightharpoonup u \text{ in } W_0^{1,p}\left(0,T;\mathbb{R}^N\right),\\&u_m\longrightarrow u \text{ in } C_0\left(0,T;\mathbb{R}^N\right).\end{aligned} \tag{166}$$

Next, we will prove that $u_m\longrightarrow u$ in $W_0^{1,p}(0,T;\mathbb{R}^N)$. By Proposition 46, it suffices to prove that the following inequality holds:

$$\lim_{m\longrightarrow\infty}\langle A(u_m)-A(u),u_m-u\rangle\leq 0. \tag{167}$$

In fact, from the choice of the sequence $\{u_m\}_{m\geq 1}$, we have

$$|\langle u_m^*,u_m\rangle|\leq\varepsilon_m\downarrow 0. \tag{168}$$

Then, by (155), we have

$$\langle A(u_m),u_m-u\rangle\rangle-\int_0^T(w_m(t),(u_m(t)-u(t)))_{\mathbb{R}^N}dt\leq\varepsilon_m,\forall m\geq 1. \tag{169}$$

By (3), $\{w_m\}\subset L^1[0,T]$ is bounded and

$$\lim_{m\longrightarrow\infty}\int_0^T(w_m(t),(u_m(t)-u(t)))_{\mathbb{R}^N}dt=0. \tag{170}$$

Then,

$$\limsup_{m\longrightarrow\infty}\langle A(u_m),u_m-u\rangle\leq 0. \tag{171}$$

So, we have

$$\limsup_{m\longrightarrow\infty}\langle A(u_m)-A(u),u_m-u\rangle\le 0,\varepsilon_m\downarrow 0. \qquad (172)$$

5. An Application

Inspired by the example result of Galewski-Rădulescu [6] (Theorem 7), we provide in this section an existence and uniqueness result for the problem

$$Ax=F(x)+\xi, \qquad (173)$$

where $\xi\in\mathbb{R}^n$ is fixed; A is an $n\times n$ matrix which does not need to be positive definite, negative definite, or symmetric; and $F:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is a locally Lipschitz function.

Theorem 47. *Let A be an $n\times n$ matrix. If $F:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is a locally Lipschitz mapping satisfying the following conditions:*

(1) *For any $\xi\in\mathbb{R}^n$, there exists a continuous nondecreasing function $h:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ such that*

$$\int_0^\infty\frac{ds}{1+h(s)}=+\infty, \qquad (174)$$

and the functional $\varphi_\xi:\mathbb{R}^n\longrightarrow\mathbb{R}$ defined by

$$\varphi_\xi(x)=\|Ax-F(x)-\xi\| \qquad (175)$$

satisfies the (h)-condition

(2) *For any $x\in\mathbb{R}^n$ and for every $T\in\partial F(x)$, $(A-T)$ is invertible*

Then, problem (173) has a unique solution for fixed $\xi\in\mathbb{R}^n$. Moreover, the map that assigns to each $\xi\in\mathbb{R}^n$, the unique solution of problem (173) is locally Lipschitz.

Proof. Consider the function $f(x)=Ax-F(x)$ from $\mathbb{R}^n$ to itself. By assumption (173) and Lemma 43, for any $\xi\in\mathbb{R}^n$, the functional $\varphi_\xi:\mathbb{R}^n\longrightarrow\mathbb{R}$ defined by

$$\varphi_\xi(x)=\frac{1}{2}\|f(x)-\xi\|^2=\frac{1}{2}\|A(x)-F(x)-\xi\|^2 \qquad (176)$$

satisfies the (h)-condition. In addition, according to (173), $\partial f(x)=A-\partial F(x)$ is of maximal rank for any $x\in\mathbb{R}^n$. Then, we achieve the proof applying Theorem 36. □

Example of matrix A and function F satisfying conditions of Theorem 47.

Let us take a matrix

$$A=\begin{pmatrix}-3 & 1\\ 2 & -1\end{pmatrix} \qquad (177)$$

and the function F defined from $\mathbb{R}^2$ to $\mathbb{R}^2$ by

$$F(u)=\left(x^3+|y|,2x+|x|+y^3\right),u=(x,y). \qquad (178)$$

Considering the Euclidean norm

$$\|u\|=\sqrt{x^2+y^2}\text{ for all }u\in\mathbb{R}^2, \qquad (179)$$

we have

$$\begin{aligned}\left\|\left(x^3,y^3\right)\right\|^2-\left(\frac{1}{2}\|u\|^3\right)^2&=x^6+y^6-\frac{1}{4}\left(x^2+y^2\right)^3\\&=\frac{3}{4}\left(x^2+y^2\right)\left(x^2-y^2\right)^2\ge 0.\end{aligned} \qquad (180)$$

It follows that

$$\left\|\left(x^3,y^3\right)\right\|\ge\frac{1}{2}\|u\|^3. \qquad (181)$$

On the other hand, for $u\in\mathbb{R}^2$,

$$\|Au\|\le\|A\|\cdot\|u\|. \qquad (182)$$

From (181) and (182), we have

$$\begin{aligned}\|F(u)-Au-\xi\|&=\left\|\left(x^3+|y|,2x+|x|+y^3\right)-Au-\xi\right\|\\&\ge\left\|\left(x^3,y^3\right)\right\|-\|(0,2x)\|-\|(|y|,|x|)\|-\|Au\|-\|\xi\|\\&\ge\frac{1}{2}\|u\|^3-2\|u\|-\|u\|-\|A\|\cdot\|u\|-\|\xi\|\\&\ge\left(\frac{1}{2}\|u\|^2-3-\|A\|\right)\|u\|-\|\xi\|.\end{aligned} \qquad (183)$$

Hence, for fixed $\xi\in\mathbb{R}^2$, the function $\varphi_\xi:\mathbb{R}^n\longrightarrow\mathbb{R}$ defined by

$$\varphi_\xi(u)=\|F(u)-Au-\xi\| \qquad (184)$$

is coercive. Consequently, the function φ_ξ satisfies the (h)-condition.

Let $u=(x,y)\in\mathbb{R}^2$.

(1) If $x\ne 0$ and $y\ne 0$, then F is differentiable at u and $\partial F(u)-A=JF(u)-A$ is defined by

$$\partial F(u)-A=\begin{pmatrix}3x^2+3 & \operatorname{sgn}(y)-1\\ \operatorname{sgn}(x) & 3y^2+1\end{pmatrix}. \qquad (185)$$

Thus, $\partial F(u)-A$ will be one of the following matrices:

$$\begin{pmatrix} 3x^2+3 & 0 \\ 1 & 3y^2+1 \end{pmatrix}, \begin{pmatrix} 3x^2+3 & -2 \\ 1 & 3y^2+1 \end{pmatrix}, \begin{pmatrix} 3x^2+3 & -2 \\ -1 & 3y^2+1 \end{pmatrix}, \begin{pmatrix} 3x^2+3 & 0 \\ -1 & 3y^2+1 \end{pmatrix}. \tag{186}$$

In all these cases, we have $\det(\partial F(u)-A)\neq 0$.

(2) If $x<0$ and $y=0$, then $\partial F(u)$ is defined by

$$\partial F(u)=\operatorname{conv}\left\{\begin{pmatrix} 3x^2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3x^2 & 1 \\ 1 & 0 \end{pmatrix}\right\}=\left\{\begin{pmatrix} 3x^2 & s \\ 1 & 0 \end{pmatrix}:\ -1\le s\le 1\right\}. \tag{187}$$

It follows that

$$\partial F(u)-A=\left\{\begin{pmatrix} 3x^2+3 & s-1 \\ -1 & 1 \end{pmatrix}:\ -1\le s\le 1\right\}. \tag{188}$$

Then, for $T\in\partial F(u)$, there exists $s\in[-1,1]$ such that

$$\det(T-A)=\left(3x^2+3\right)+(s-1)=x^2+s+2\ge x^2+1>0. \tag{189}$$

(3) If $x>0$ and $y=0$, then $\partial F(u)$ is the following:

$$\partial F(u)=\operatorname{conv}\left\{\begin{pmatrix} 3x^2 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 3x^2 & -1 \\ 3 & 0 \end{pmatrix}\right\}=\left\{\begin{pmatrix} 3x^2 & s \\ 3 & 0 \end{pmatrix}:\ -1\le s\le 1\right\}. \tag{190}$$

It follows that

$$\partial F(u)-A=\left\{\begin{pmatrix} 3x^2+3 & s-1 \\ 1 & 1 \end{pmatrix}:\ -1\le s\le 1\right\}. \tag{191}$$

Then, for $T\in\partial F(u)$, there exists $s\in[-1,1]$ such that

$$\det(T-A)=\left(3x^2+3\right)-(s-1)=\left(3x^2+1\right)+(3-s)\ge 3x^2+1>0. \tag{192}$$

(4) If $x=0$ and $y<0$, then $\partial F(u)$ is the following:

$$\partial F(u)=\operatorname{conv}\left\{\begin{pmatrix} 0 & -1 \\ 3 & 3y^2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 3y^2 \end{pmatrix}\right\}=\left\{\begin{pmatrix} 0 & -1 \\ \lambda & 3y^2 \end{pmatrix}:\ 1\le\lambda\le 3\right\}. \tag{193}$$

It follows that

$$\partial F(u)-A=\left\{\begin{pmatrix} 3 & -2 \\ \lambda-2 & 3y^2+1 \end{pmatrix}:\ 1\le\lambda\le 3\right\}. \tag{194}$$

Then, for $T\in\partial F(u)$, there exists $\lambda\in[1,3]$ such that

$$\det(T-A)=3\left(3y^2+1\right)+2(\lambda-2)=9x^2+2\lambda-1>0. \tag{195}$$

(5) If $x=0$ and $y>0$, then $\partial F(u)$ is the following:

$$\partial F(u)=\operatorname{conv}\left\{\begin{pmatrix} 0 & 1 \\ 3 & 3y^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 3y^2 \end{pmatrix}\right\}=\left\{\begin{pmatrix} 0 & 1 \\ \lambda & 3y^2 \end{pmatrix}:\ 1\le\lambda\le 3\right\}. \tag{196}$$

It follows that

$$\partial F(u)-A=\left\{\begin{pmatrix} 3 & 0 \\ \lambda-2 & 3y^2+1 \end{pmatrix}:\ 1\le\lambda\le 3\right\}. \tag{197}$$

Then, for $T\in\partial F(u)$, there exists $\lambda\in[1,3]$ such that

$$\det(T-A)=3\left(3y^2+1\right)>0. \tag{198}$$

(6) If $u=(0,0)$, then $\partial F(u)$ is the following:

$$\begin{aligned}\partial F(u)&=\operatorname{conv}\left\{\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}\\ &=\left\{\begin{pmatrix} 0 & \tau \\ s & 0 \end{pmatrix}:\ (\tau,s)\in[-1,1]\times[1,3]\right\}.\end{aligned} \tag{199}$$

It follows that

$$\partial F(0,0)-A=\left\{\begin{pmatrix} 3 & \tau-1 \\ s-2 & 1 \end{pmatrix}:\ (\tau,s)\in[-1,1]\times[1,3]\right\}. \tag{200}$$

Then, for $T\in\partial F(u)$, there exists $(\tau,s)\in[-1,1]\times[1,3]$ such that

$$\det(T-A)=3-(s-2)(\tau-1)=(s-1)(1-\tau)+\tau+2\ge\tau+2>0. \tag{201}$$

6. Conclusion

We have provided a general nonsmooth global implicit function theorem that yields Galewski-Rădulescu's nonsmooth global implicit function theorem and a series of results on the existence, uniqueness, and possible continuity of global implicit functions for the zeros of locally Lipschitz functions. Our results deal with functions defined on infinite dimensional Banach spaces and thus generalize also classical Clarke's implicit function theorem for functions $F : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ by replacing $\mathbb{R}^p$ by any Banach space Y. We have worked in this paper under the (h)-condition which is weaker than the coercivity required in [6]. Our method is based on a variational approach and a recent nonsmooth version of Mountain Pass Theorem.

More precisely, firstly, we have proved our Theorem 31 on the existence and uniqueness of the global implicit function theorem for equations $F(x, y) = 0$, where $F : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^n$ is a locally Lipschitz function with Y a Banach space. Secondly, we observe that this extension to infinite dimension may not guarantee the continuity of the global implicit function. Thus, we provide an additional hypothesis on Theorem 31 in order to obtain the continuity of the implicit function f. Moreover, our Lemmas 42 and 43 allow us to prove other more general results on the existence and uniqueness of global implicit functions under the (h)-condition on the function $x \mapsto \|F(x, y)\|^\alpha$ with $0 < \alpha < 2$.

Data Availability

The academic material resources we use are in the form of papers which are all deposited by their respective authors and have Digital Object Identifiers (DOI). These Digital Object Identifiers (DOI) of our reference papers are given in a supporting material sent to the Editorial Office.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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