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Polynomial Operator in the Shifts in Discrete Algebraic Dynamical Systems

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Abstract

The vector space of the multi-indexed sequences over a field and the vector space of the sequences with finite support are dual to each other, with respect to an appropriate scalar product. It follows that the *polynomial operator in the shift* which U. Oberst and J. C. Willems have introduced to define time invariant discrete linear dynamical systems can be explained as the adjoint of the polynomial multiplication.

Keywords: Dynamical system; behavior; polynomial operator in the shift; scalar product; adjoint of a linear mapping; categories and functors.

2010 Mathematics Subject Classification: 13C60, 15A04, 39A10, 47A05

1 Introduction

Discrete algebraic dynamical systems theory essentially studies subsets $\mathcal{B}$ (called *behavior*) of the set of functions from a time set $\mathbb{T}$ (usually $\mathbb{N}^r, \mathbb{Z}^r$) to $\mathbb{F}^l$, where $\mathbb{F}$ is a field. In this paper, we will

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use the time set $\mathbb{N}^r$ only, for simplicity. For reference, we use the basic papers of Jan C. Willems and Ulrich Oberst who have established the theory: [1, 2] and [3]. For more recent developments on the subject, interested readers may consult [4, 5, 6, 7, 8, 9, 10]. In [1, 2], where the case $r = 1$ is treated, these subsets are required to be linear, time invariant, closed with respect to the topology of pointwise convergence. In [3], the general case $r \geqslant 1$ has been treated and algebraic structures used to interpret these properties.

A key concept in defining discrete algebraic linear systems is the *polynomial operator in the shift*. There is, in general, no much algebraic explanation about the construction of this operator in the texts about discrete algebraic dynamical systems. An interpretation, in [3], which is of course true, uses systems defined by the polynomial operator in the shifts itself but this does not explain its origin because an object cannot be explained by referencing to itself.

Our goal is to give an explanation of this operator, parting from initial notions which do not depend on it. We believe our approach is simpler and more natural. This gives us a deeper understanding of this operator and allows to reconnect with classical algebraic structures, giving discrete algebraic dynamical systems an elegant aspect.

2 Discrete Algebraic Dynamical Systems

2.1 Notations

Let $\mathbb{N}$ be the set of the natural integers , $\mathbb{F}$ a commutative field and $r \geqslant 1$ an integer. The set of the multi-indexed sequences

$$\begin{aligned} W : \mathbb{N}^r &\longrightarrow \mathbb{F} \\ W &\longmapsto W(\alpha) = W_\alpha \end{aligned}$$

is denoted by $\mathbb{F}^{\mathbb{N}^r}$. It is an $\mathbb{F}$-vector space. For $W \in \mathbb{F}^{\mathbb{N}^r}$, the *support* of W is the set

$$\text{Supp}(W) = \{\alpha \in \mathbb{N}^r \mid W_\alpha \neq 0\}.$$

The subset of $\mathbb{F}^{\mathbb{N}^r}$ with finite support is denoted by $\mathbb{F}^{(\mathbb{N}^r)}$; it is an $\mathbb{F}$ vector subspace of $\mathbb{F}^{\mathbb{N}^r}$.

For $\rho = 1, \dots, r$, let X_ρ (resp. Y_ρ) be letters, called also *variables*. For simplicity, X (resp. Y) will denote $X_1, \dots, X_r$ (resp $Y_1, \dots, Y_r$) and for $\alpha \in \mathbb{N}^r$ we define X^α (resp. Y^α) by

$$X^\alpha = X_1^{\alpha_1} \cdots X_r^{\alpha_r} \quad \text{(resp. } Y^\alpha = Y_1^{\alpha_1} \cdots Y_r^{\alpha_r}\text{)}.$$

For $\alpha \in \mathbb{N}^r$, let δ_α be the mapping

$$\begin{aligned} \delta_\alpha : \mathbb{N}^r &\longrightarrow \mathbb{F} \\ \beta &\longmapsto \delta_\alpha(\beta) = \begin{cases} 0, & \text{if} \quad \alpha \neq \beta, \\ 1, & \text{if} \quad \alpha = \beta. \end{cases} \end{aligned} \tag{2.1}$$

Then $\delta_\alpha \in \mathbb{F}^{(\mathbb{N}^r)}$ with $\text{Supp}(\delta_\alpha) = \{\alpha\}$.

Let $\mathbf{D} = \mathbb{F}[X_1, \dots, X_r] = \mathbb{F}[X]$ be the $\mathbb{F}$-vector space of the polynomials with the r variables $X_1, \dots, X_r$ and $\mathbf{A} = \mathbb{F}[[Y_1, \dots, Y_r]] = \mathbb{F}[[Y]]$ that of the formal power series with the r variables $Y_1, \dots, Y_r$. The family $(X^\alpha)_{\alpha \in \mathbb{N}^r}$ is an $\mathbb{F}$-base of $\mathbf{D}$, thus an element of $\mathbf{D}$ can be written uniquely as

$$d(X) = \sum_{\alpha \in \mathbb{N}^r} d_\alpha X^\alpha \quad \text{with} \quad d_\alpha \in \mathbb{F} \quad \text{for all } \alpha \in \mathbb{N}^r,$$

where $d_\alpha = 0$ except for a finite number of α. An element $W(Y)$ of $\mathbf{A}$ can be uniquely expressed as

$$W(Y) = \sum_{\alpha \in \mathbb{N}^r} W_\alpha Y^\alpha \quad \text{with} \quad d_\alpha \in \mathbb{F} \quad \text{for all} \quad \alpha \in \mathbb{N}^r .$$

Therefore, we get the $\mathbb{F}$-vector spaces isomorphisms

$$\begin{aligned}\mathbf{D} = \mathbb{F}[X_1, \dots, X_r] &\cong \mathbb{F}^{(\mathbb{N}^r)} \\ X^\alpha &\longleftrightarrow \delta_\alpha\end{aligned}$$

and

$$\begin{aligned}\mathbf{A} = \mathbb{F}[[Y_1, \dots, Y_r]] &\cong \mathbb{F}^{\mathbb{N}^r} \\ \sum_{\alpha\in\mathbb{N}^r} W_\alpha Y^\alpha &\longleftrightarrow \sum_{\alpha\in\mathbb{N}^r} W_\alpha \delta_\alpha.\end{aligned}$$

By these isomorphisms, we may identify X^α (resp. Y^α) with the element δ_α of $\mathbb{F}^{(\mathbb{N}^r)}$ (resp. of $\mathbb{F}^{\mathbb{N}^r}$). If $W \in \mathbb{F}^{\mathbb{N}^r}$, we may write $W = (W_\alpha)_{\alpha\in\mathbb{N}^r}$, where $W_\alpha = W(\alpha)$ for all $\alpha \in \mathbb{N}^r$. Finally, we may write the following identifications

$$W = (W_\alpha)_{\alpha\in\mathbb{N}^r} = \sum_{\alpha\in\mathbb{N}^r} W_\alpha Y^\alpha = W(Y).$$

The set $\mathbb{F}^{\mathbb{N}^r}$ (resp. $\mathbb{F}^{(\mathbb{N}^r)}$) is also denoted by $\mathbf{A}$ (resp. $\mathbf{D}$). Let $k, l \geqslant 1$ be integers. The cartesian product $\mathbf{A} \times \cdots \times \mathbf{A}$ (resp. $\mathbf{D} \times \ldots \times \mathbf{D}$) ($l$ times) is denoted by $\mathbf{A}^l$ (resp. $\mathbf{D}^l$). The set of matrices with k lines and l columns with coefficients in $\mathbf{A}$ (resp. in $\mathbf{D}$) is denoted $\mathbf{A}^{k,l}$ (resp. $\mathbf{D}^{k,l}$). Denoting the variables $X_1, \dots, X_r$ simply by X, an element $R(X) \in \mathbf{D}^{k,l}$ is of the form

$$R(X) = (R_{\kappa\lambda}(X))_{1\leqslant\kappa\leqslant k, 1\leqslant\lambda\leqslant\ l}$$

where $R_{\kappa\lambda}(X) \in \mathbf{D}$ for $\kappa = 1, \dots, k$ and $\lambda = 1, \dots, l$. For an $\mathbb{F}$-vector space V, the $\mathbb{F}$-vector space of the linear forms $f : V \longrightarrow \mathbb{F}$ is denoted by $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$.

2.2 Discrete linear dynamical systems according to Oberst

Now, we are going to give the definition of an algebraic discrete dynamical systems as formulated in [3] (case $r \geqslant 1$) , which is the generalization of those of Willems (case $r = 1$).

For a polynomial $P(X) = \sum_{\alpha\in\mathbb{N}^r} P_\alpha X^\alpha \in \mathbf{D}$ and an element $W \in \mathbf{A}$, the element $P(X)W$ of $\mathbf{A}$ is defined by

$$(P(X)W)_\beta = \sum_{\alpha\in\mathbb{N}^r} P_\alpha W_{\alpha+\beta} \tag{2.2}$$

for all $\beta \in \mathbb{N}^r$.

For a polynomial matrix $R(X) \in \mathbf{D}^{k,l}$ and a vector $W \in \mathbf{A}^l$, the element $R(X)W \in \mathbf{A}^k$ is defined by

$$R(X)W = (\sum_{\lambda=1}^{l} R_{\kappa\lambda}(X)W_\lambda(Y))_{\kappa=1,\dots,k}. \tag{2.3}$$

Definition 2.1. A *discrete linear dynamical system* (or simply a *system*) is a subset $\mathcal{B}$ of $\mathbf{A}^l$ of the form

$$\mathcal{B} = \{W \in \mathbf{A}^l \mid R(X)W = 0\}. \tag{2.4}$$

We call these systems *discrete algebraic dynamical systems* because the properties of *time invariance, linearity and completeness* (or closure with respect to the topology of pointwise convergence), ([1]) is captured in the algebraic structure of $\mathcal{B}$, more precisely because of the fact that $\mathcal{B}$ is a $\mathbf{D}$-submodule of $\mathbf{A}^l$.

This definition is based on the equation (2.3), which uses (2.2). So, the question is: what does (2.2) mean ? Where does it come from and how can it be algebraically explained ?

Let get a closer look at (2.2). When $P(X) = X^\alpha$ is a monic monomial, we get the element $X^\alpha W \in \mathbf{A}^l$ which verifies

$$(X^\alpha W)_\beta = W_{\alpha+\beta},$$

i.e. the value of $X^\alpha W$ at β is the value of W at $\alpha + \beta$ for all $\beta \in \mathbb{N}^r$. For the case $r = 1$, the values of the power series $X^n W$ are those of W, but "shifted" to the left. For this reason, the action of X^α on W is called *(left) shift(s)* and X^α a *shift(s) operator*, the number of shifts being the integer r. By linearity, every polynomial $P(X) \in \mathbf{D}$ defines, by equation (2.2), a *polynomial operator in the shift(s)*; it is the linear mapping of $\mathbf{D}$-modules, denoted again by $P(X)$ such that

$$\begin{aligned} P(X) : \mathbf{A} &\longrightarrow \mathbf{A} \\ W &\longmapsto P(X)W. \end{aligned}$$

We remark that (2.3) is very similar to the "product" of the matrix $R(X)$ and the vector W in the sense that the term $\sum_{\lambda=1}^{l} R_{\kappa\lambda}(X)W_\lambda(Y)$ gives the κ-th row, like in the ordinary matrix-vector product. Therefore, we may think $R(X)W$ as a product of the matrix $R(X)$ and the vector W, constructed from the polynomial-vector multiplication (2.2) which we are going to explain.

In [3, p.20], the category $\mathrm{Modf}(\mathbf{D})$ of the finitely generated $\mathbf{D}$ modules and the category $\mathbf{Syst}(\mathbf{A})$ of all systems which are subsets of $\mathbf{A}^l$ for some integer $l \geqslant 1$ are introduced. Then, using the *contravariant functor*

$$\begin{aligned} \boldsymbol{\mathcal{S}} = \mathrm{Hom}_{\mathbf{D}}(-, \mathbf{A}) : \mathrm{Modf}(\mathbf{D})^{\mathrm{op}} &\longrightarrow \mathbf{Syst}(\mathbf{A}) \\ M &\longmapsto \mathrm{Hom}_{\mathbf{D}}(M, \mathbf{A}) \\ (f : M \longrightarrow N) &\longmapsto \left\{ \begin{aligned} \mathrm{Hom}_{\mathbf{D}}(f, \mathbf{A}) : \mathrm{Hom}_{\mathbb{F}}(N, \quad \mathbf{A}) &\longrightarrow \mathrm{Hom}_{\mathbb{F}}(M, \mathbf{A}) \\ u &\longmapsto u \circ f, \end{aligned} \right. \end{aligned}$$

where $\mathrm{Modf}(\mathbf{D})^{\mathrm{op}}$ is the *opposite category* of $\mathrm{Modf}(\mathbf{D})$ ([11]), it is shown that the $\mathbf{D}$-linear mapping of $\mathbf{D}$-modules, denoted by R,

$$\begin{aligned} R : \mathbf{A}^l &\longrightarrow \mathbf{A}^k \\ W &\longmapsto R(X)W \end{aligned}$$

is the image under $\boldsymbol{\mathcal{S}}$ of the $\mathbf{D}$-linear mapping of $\mathbf{D}$-modules defined by the right polynomial multiplication by $R(X)$, denoted by R^T :

$$\begin{aligned} R^T : \mathbf{D}^k &\longrightarrow \mathbf{D}^l \\ d(X) &\longmapsto d(X) \cdot R(X), \end{aligned}$$

i.e $\boldsymbol{\mathcal{S}}(R^T) = R$. In other terms, the action of the matrix $R(X)$ on W, which is $R(X)W$, defined by the (2.3) comes from the transformation by $\boldsymbol{\mathcal{S}}$ of the right polynomial matrix multiplication by $R(X)$. The problem with this explanation is that the functor $\boldsymbol{\mathcal{S}}$ is defined between two categories, one of which being defined from systems, which are already defined using (2.3).

In this paper, we use a modified approach: we consider the category $\mathrm{Vect}(\mathbb{F})$ of the vector spaces over $\mathbb{F}$ and use the functor

$$\begin{aligned} \mathrm{Hom}_{\mathbb{F}}(-, \mathbb{F}) : \mathrm{Vect}(\mathbb{F}) &\longrightarrow \mathrm{Vect}(\mathbb{F}) \\ E &\longmapsto \mathrm{Hom}_{\mathbb{F}}(E, \mathbb{F}) \\ (f : E \longrightarrow F) &\longmapsto \left\{ \begin{aligned} \mathrm{Hom}_{\mathbb{F}}(f, \mathbb{F}) : \mathrm{Hom}_{\mathbb{F}}(F, \quad \mathbb{F}) &\longrightarrow \mathrm{Hom}_{\mathbb{F}}(E, \mathbb{F}) \\ u &\longmapsto u \circ f. \end{aligned} \right. \end{aligned} \tag{2.5}$$

If for $E, F \in \mathrm{Vect}(\mathbb{F})$, there exists a *scalar product* $\langle -, - \rangle : E \times F \longrightarrow \mathbb{F}$, then the linear mapping $u \circ f$ in (2.5) coincides with the classical *adjoint* (without the use of categories and functor) of the linear mapping f.

3 Main Theorem

3.1 Scalar product

In order to make an effective use of (2.5), we shall construct a scalar product $\langle -,-\rangle : \mathbf{D}\times\mathbf{A}\longrightarrow\mathbb{F}$.

Proposition 3.1. *The bilinear mapping*

$$\begin{aligned}&\langle -,-\rangle : \mathbf{D}\times\mathbf{A}\longrightarrow\mathbb{F}\\ &(d(X),W(Y))\longmapsto\langle d(X),W(Y)\rangle=\sum_{\alpha\in\mathbb{N}^r} d_\alpha\cdot W_\alpha\end{aligned}\tag{3.1}$$

satisfies the following properties:
(1) The homomorphism

$$\begin{aligned}\mathbf{D}&\longrightarrow \mathrm{Hom}_{\mathbb{F}}(\mathbf{A},\mathbb{F})\\ d(X)&\longmapsto\left\{\begin{array}{ll}\langle d(X),-\rangle : & \mathbf{A}\longrightarrow\mathbb{F}\\ & W(Y)\longmapsto\langle d(X),W(Y)\rangle\end{array}\right.\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}\mathbf{A}&\longrightarrow \mathrm{Hom}_{\mathbb{F}}(\mathbf{D},\mathbb{F})\\ W(Y)&\longmapsto\left\{\begin{array}{ll}\langle -,W(Y)\rangle : & \mathbf{D}\longrightarrow\mathbb{F}\\ & d(X)\longmapsto\langle d(X),W(Y)\rangle\end{array}\right.\end{aligned}\tag{3.3}$$

are injective .
(2) The monomorphism (3.3) *is an isomorphism of vector spaces.*

Proof. (1) Let $d_1(X),d_2(X)\in\mathbf{D}$ such that $\langle d_1(X),W(Y)\rangle=\langle d_2(X),W(Y)\rangle$ for all $W(Y)\in\mathbf{A}$. for each $\alpha\in\mathbb{N}^r$, set $W(Y)=\delta_\alpha$ (see (2.1)). It follows that $\langle d_1(X),\delta_\alpha\rangle=d_{1\alpha}$ and $\langle d_2(X),\delta_\alpha\rangle=d_{2\alpha}$ i.e. $d_1(X)=d_2(X)$. Thus the homomorphism (3.2) is injective.
Let $W_1(Y),W_2(Y)\in\mathbf{A}$ such that $\langle -,W_1(Y)\rangle=\langle -,W_2(Y)\rangle$. Then $\langle d(X),W_1(Y)\rangle=\langle d(X),W_2(Y)\rangle$ for all $d(X)\in\mathbf{D}$. Setting $d(X)=\delta_\alpha$ for each $\alpha\in\mathbb{N}^r$, we get $(W_1)_\alpha=(W_2)_\alpha$ i.e $W_1=W_2$. Thus (3.3) is injective.
(2) Let $\psi\in\mathbf{Hom}_{\mathbb{F}}(\mathbf{A},\mathbb{F})$ and $W(Y)\in\mathbf{A}$ defined by $W_\alpha=\psi(\delta_\alpha)$ for all $\alpha\in\omega$. For each $d(X)\in\mathbf{D}$ we have

$$d(X)=\sum_{\alpha\in\mathbb{N}^r} d_\alpha\delta_\alpha$$

and

$$\psi(d(X))=\psi(\sum_{\alpha\in\mathbb{N}^r} d_\alpha\delta_\alpha)=\sum_{\alpha\in\mathbb{N}^r} d_\alpha\psi(\delta_\alpha)=\sum_{\alpha\in\mathbb{N}^r} d_\alpha W_\alpha=\langle d(X),W(Y)\rangle.$$

Therefore $\psi=\langle -,W(Y)\rangle$. Thus the monomorphism (3.3) is surjective, i.e. a vector spaces isomorphism. □

We say that $\langle -,-\rangle$ is a scalar product and the vector spaces $\mathbf{A}$ and $\mathbf{D}$ are *dual* (to each other).

3.2 Polynomial operator in the shift(s)

Here we prove our main theorem 3.3.

Part (2) of proposition 3.1 gives the isomorphism

$$\mathbf{A}\cong\mathrm{Hom}_{\mathbb{F}}(\mathbf{D},\mathbb{F}),\ W(Y)\longmapsto\langle -,W(Y)\rangle.$$

Considering $W \in \mathbf{A}$ as element of $\mathrm{Hom}_{\mathbb{F}}(\mathbf{D}, \mathbb{F})$, we have

$$\langle d(X), W \rangle = W(d(X)). \tag{3.4}$$

In other terms, with the identification $\mathbf{A} = \mathrm{Hom}_{\mathbb{F}}(\mathbf{D}, \mathbb{F})$, we may view W as acting on X^α by

$$W(X^\alpha) = W(\alpha) = W_\alpha. \tag{3.5}$$

Let $\mathrm{Vect}(\mathbb{F})$ be the category whose object consists of all $\mathbb{F}$-vector spaces and for two objects E, F of $\mathrm{Vect}(\mathbb{F})$, the set of morphism from E to F is $\mathrm{Hom}_{\mathbb{F}}(E, F)$, which consists of all linear mappings from E to F. We then have the covariant functor $\mathrm{Hom}_{\mathbb{F}}(-, \mathbb{F})$ defined by

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{F}}(-, \mathbb{F}) : \mathrm{Vect}(\mathbb{F}) &\longrightarrow \mathrm{Vect}(\mathbb{F}) \\
E &\longmapsto \mathrm{Hom}_{\mathbb{F}}(E, \mathbb{F}) \\
(f : E \longrightarrow F) &\longmapsto \left\{ \begin{aligned} \mathrm{Hom}_{\mathbb{F}}(f, \mathbb{F}) : \mathrm{Hom}_{\mathbb{F}}(F, \quad \mathbb{F}) &\longrightarrow \mathrm{Hom}_{\mathbb{F}}(E, \mathbb{F}) \\ u &\longmapsto u \circ f, \end{aligned} \right.
\end{aligned}$$

([11]). The following definition is in the same book :

Definition 3.1. Let $E, F \in \mathrm{Vect}(\mathbb{F})$ and $f \in \mathrm{Hom}_{\mathbb{F}}(E, F)$. The *adjoint* of f is the linear mapping $\mathrm{Hom}_{\mathbb{F}}(f, \mathbb{F})$.

Now we are going to look at the adjoints of particular linear mappings: take $E = F = \mathbf{D}$ and fix $d(X) \in \mathbf{D}$. We get the " multiplication by $d(X)$ ", which is the linear mapping

$$d(X) : \mathbf{D} \longrightarrow \mathbf{D},\ c(X) \longmapsto c(X) \cdot d(X)$$

which we also denoted by $d(X)$. For the case $d(X) = X^\beta$ and $\beta \in \mathbb{N}^r$. We get the " multiplication by X^β ":

$$X^\beta : \mathbf{D} \longrightarrow \mathbf{D},\ c(X) \longmapsto c(X) \cdot X^\beta.$$

The adjoint of the multiplication by X^β is given by the following lemma:

Lemma 3.1. *The adjoint of the multiplication by X^β*

$$\begin{aligned} X^\beta : \mathbf{D} &\longrightarrow \mathbf{D} \\ c(X) &\longmapsto c(X) \cdot X^\beta, \end{aligned}$$

is the $\mathbb{F}$-endomorphism

$$\begin{aligned} \mathbf{A} &\longrightarrow \mathbf{A} \\ W(Y) = \sum_{\alpha \in \mathbb{N}^r} W_\alpha Y^\alpha &\longmapsto \sum_{\alpha \in \mathbb{N}^r} W_{\alpha+\beta} Y^\alpha. \end{aligned} \tag{3.6}$$

Proof. We already know that $\mathrm{Hom}_{\mathbb{F}}(\mathbf{D}, \mathbb{F}) = \mathbf{A}$. If $W \in \mathbf{A}$, the mapping $\mathrm{Hom}_{\mathbb{F}}(X^\beta, \mathbb{F})(W) = W \circ X^\beta$ is an element of $\mathbf{A}$ and from (3.5), we have that

$$(W \circ X^\beta)(\alpha) = W \circ X^\beta(X^\alpha) = W(X^\beta \cdot X^\alpha) = W(X^{\alpha+\beta}) = W_{\alpha+\beta}$$

for all $\alpha \in \mathbb{N}^r$ (the symbol $\circ$ is the composition of mappings). This completes the proof of (3.6). □

The adjoint of the multiplication by X^β is then the shift(s) operator. We use symbol "$\circ$ " to mean that X^β operates on a power series. Thus,

$$X^\beta \circ W(Y) = \mathrm{Hom}_{\mathbb{F}}(X^\beta, \mathbb{F})(W(Y)) = \sum_{\alpha \in \mathbb{N}^r} W_{\alpha+\beta} Y^\alpha \in \mathbf{A}\,.$$

Example 3.2. *Fix $\alpha \in \mathbb{N}^r$ and take $W(Y) = Y^\alpha$. For each $\beta \in \mathbb{N}^r$, we have*

$$X^\beta \circ Y^\alpha = \begin{cases} Y^{\alpha-\beta} & \text{if} \quad \beta \leqslant_+ \alpha, \\ 0 & \text{otherwise}, \end{cases}$$

where $\beta \leqslant_+ \alpha$ means that $\beta_i \leqslant \alpha_i$ for $i = 1, \dots, r$.

We have the following fundamental property:

$$(X^\alpha \cdot X^\beta) \circ W(Y) = X^\alpha \circ (X^\beta \circ W(Y)) \tag{3.7}$$

for all $\alpha, \beta \in \mathbb{N}^r$ and $W(Y) \in \mathbf{A}$.

Now consider the general case of the polynomial multiplication by $d(X) \in \mathbf{D}$:

$$d(X) : \mathbf{D} \longrightarrow \mathbf{D}, \;\; c(X) \longmapsto c(X) \cdot d(X). \tag{3.8}$$

If $d(X) = \sum_\beta d_\beta X^\beta$, we may view $d(X)$ as a linear combination of X^β. Taking the adjoint, we have

$$\mathrm{Hom}_{\mathbb{F}}(d(X), \mathbb{F}) = \sum_\beta d_\beta \, \mathrm{Hom}_{\mathbb{F}}(X^\beta, \mathbb{F}),$$

and using (3.6), we have

$$\begin{aligned} \mathrm{Hom}_{\mathbb{F}}(d(X), \mathbb{F})(W(Y)) &= \sum_\beta d_\beta \, \mathrm{Hom}_{\mathbb{F}}(X^\beta, \mathbb{F})(W(Y)) \\ &= \sum_\alpha (\sum_\beta (d_\beta W_{\alpha+\beta}) Y^\alpha. \end{aligned}$$

We get the polynomial operator in the shift(s) defined by $d(X)$, as in (2.2). We have thus proved the following theorem :

Theorem 3.3. *The adjoint of the polynomial multiplication by $d(X)$*

$$\begin{aligned} d(X) : \mathbf{D} &\longrightarrow \mathbf{D} \\ c(X) &\longmapsto c(X) \cdot d(X), \end{aligned}$$

is the polynomial operator in the shift(s), also denoted by $d(X)$ and defined as

$$\begin{aligned} d(X) : \mathbf{A} &\longrightarrow \mathbf{A} \\ W(Y) &\longmapsto d(X) \circ W(Y) = \sum_\alpha (\sum_\beta d_\beta W_{\alpha+\beta}) Y^\alpha. \end{aligned}$$

Using (3.7), the proof of the following proposition is left to the reader:

Proposition 3.2. *The operation*

$$\begin{aligned} \mathbf{D} \times \mathbf{A} &\longrightarrow \mathbf{A} \\ (d(X), W(X)) &\longmapsto d(X) \circ W(Y) \end{aligned}$$

is an external operation of $\mathbf{D}$ on $\mathbf{A}$. It provides $\mathbf{A}$, and therefore $\mathbf{A}^l$ (where $l \geqslant 1$ is an integer) with a $\mathbf{D}$-module structure.

We call this external operation *circle* or the *circle multiplication*. With this multiplication, we can define the multiplication of a polynomial matrix $R(X) \in \mathbf{D}^{k,l}$ and a power series vector $W(Y) \in \mathbf{A}^l$ by

$$R(X) \circ W(Y) = \begin{pmatrix} R_1(X) \circ W(Y) \\ \vdots \\ R_k(X) \circ W(Y) \end{pmatrix} = \begin{pmatrix} \sum_{\lambda=1}^{l} R_{1\lambda}(X) \circ W_\lambda(Y) \\ \vdots \\ \sum_{\lambda=1}^{l} R_{k\lambda}(X) \circ W_\lambda(Y) \end{pmatrix} \in \mathbf{A}^k .$$

with $R_\kappa(X) = (R_{\kappa 1}(X), \ldots, R_{\kappa l}(X)) \in \mathbf{D}^l$ being the κ-th row of $R(X)$, for $\kappa = 1, \ldots, k$ and $W_\lambda(Y)$ the λ-th row of $W(Y)$ for $\lambda = 1, \ldots, l$. This is a new notation for (2.3). One can then prove the following corollaries ([3]):

Corollary 3.4. *The mapping*

$$\begin{aligned} \circ : \mathbf{D}^{k,l} \times \mathbf{A}^l &\longrightarrow \mathbf{A}^k \\ (R(X), W(Y)) &\longmapsto R(X) \circ W(Y). \end{aligned} \tag{3.9}$$

(again denoted by $\circ$) is $\mathbf{D}$- bilinear.

Corollary 3.5. *With the notations in* (3.9)*, fix $R(X) \in \mathbf{A}^{k,l}$. Then we get the $\mathbf{D}$-linear mapping of $\mathbf{D}$-modules (denoted again by $R(X)$), called the right multiplication by $R(X)$ in $\mathbf{A}^l$:*

$$\begin{aligned} R(X) : \mathbf{A}^l &\longrightarrow \mathbf{A}^k \\ W &\longmapsto R(X) \circ W(Y). \end{aligned} \tag{3.10}$$

Using our notations, a discrete algebraic dynamical system is then a subset $\boldsymbol{\mathcal{B}}$ of $\mathbf{A}^l$ the form

$$\boldsymbol{\mathcal{B}} = \{W(Y) \in \mathbf{A}^l \mid R(X) \circ W(Y) = 0\},$$

where $R(X) \in \mathbf{D}^{k,l}$. In other terms, it is the kernel of the linear mapping (3.10): $\boldsymbol{\mathcal{B}} = \ker R(X)$. As a kernel of a modules homomorphism, its is a $\mathbf{D}$-submodule of $\mathbf{A}^l$ and we come back to the definition 2.1 of a system.

4 Conclusions

a In the section 2, we have given a brief survey on the definition of dynamical systems and the polynomial operator in the shifts.

b In the subsection 3.1, we have formulated the scalar product and the duality of the vector spaces of multivariate polynomials and the vector space of the power series with the same number of variables (the two sets of variables are denoted differently).

c In the subsection 3.2, we have interpreted of the polynomial operator in the shifts on the power series, as the adjoint of the polynomial multiplication.

Possible future works are to prove whether the adjoint of the right multiplication by the polynomial matrix $R(X)$, denoted by $R(X)^T$ in [3] :

$$\begin{aligned} R(X)^T : \mathbf{D}^k &\longrightarrow \mathbf{D}^l \\ c(X) &\longmapsto c(X) \cdot R(X) \end{aligned}$$

is the left multiplication (3.10) by $R(X)$ in $\mathbf{A}^l$:

$$\begin{aligned} R(X) : \mathbf{A}^l &\longrightarrow \mathbf{A}^k \\ W &\longmapsto R(X) \circ W(Y) \end{aligned}$$

or not and study the case of the time set $\mathbb{Z}^r$ which involve polynomials in $X_1^{-1}, \ldots, X_r^{-1}$.

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Competing Interests

The authors declare that no competing interests exist.

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