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Nontrivial Weak Solutions of a Quasilinear Equation Involving p-Laplace Operator

Mustafa Avci[1*]

[1] *Batman University, Faculty of Economics and Administrative Sciences, Turkey.*

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Abstract

In the present paper, using direct variational approach, and the monotone operator method, the existence of nontrivial solutions for a quasilinear elliptic equation involving the p-Laplace operator is obtained.

Keywords: p-Laplace operator; variational approach; monotone operator method; quasilinear elliptic equation; Sobolev spaces.

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1 Introduction

In the present paper, we deal withe the existence and nontrivial solutions of the Dirichlet boundary problem

$$-\Delta_p u + |u|^{p-2}u = f(x,u) \;\; in\ \Omega \tag{1.1}$$

$$u = 0 \;\; on\ \partial\Omega \tag{1.2}$$

**Correspondance: avcixmustafa@gmail.com*

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $1 < p < N$.
The problems of type $(\mathbf{1.1})$-$(\mathbf{1.2})$ are very important in applications mentioned below. These type problems arises from the existence of the p-Laplacian or the p-Laplace operator $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$ which turns to the usual Laplace operator $\triangle_2 = \triangle$ for the case $p = 2$. However, in case $p \neq 2$ the situation is very crucial, as for example, one encounters the lack of the Hilbert structure of the Sobolev space $W_0^{1,p}(\Omega)$.
The problems involving the p-Laplace operator have been investigated for the last two decades intensively in different areas of applied mathematics and physics. For example, in the study of non-Newtonian fluids, nonlinear elasticity, and reaction diffusions. For the detailed background, see, for example, [1, 2] and references therein.
One of the most widely used tool for solving problems of type $(\mathbf{1.1})$-$(\mathbf{1.2})$ is the Mountain-Pass theorem. When applying this theorem, one usually needs that the functional corresponding to the related problem must have the Palais-Smale property. One way to ensure this is to assume that f satisfies some Ambrosetti-Rabinowitz-type condition [3, 4].
In the present paper, we study problem $(\mathbf{1.1})$-$(\mathbf{1.2})$ in two different approaches. First, we use variational approach and apply some specific assumptions on the nonlinearities f instead of the classical conditions, such as Ambrosetti-Rabinowitz condition. Moreover, the first eigenvalue of $p-$Laplace operator has a key role to get the first result related to problem $(\mathbf{1.1})$-$(\mathbf{1.2})$. Then, we proceed with monotone operator method to get the second result. Not using the classical tools mentioned above and considering two different approaches for the same problems to get the existence results are the difference of the present paper from the previous studies.
We continue to recall some necessary information needed through the paper. The Lebesgue spaces of measurable functions $L^p(\Omega)$, for $1 \leq p \leq \infty$, are defined by

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}\ measurable, \int_\Omega |u|^p dx < \infty\},$$

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}\ measurable, esssup_{x\in\Omega}|u(x)| < \infty\}.$$

Let define the norms

$$|u|_p = \left(\int_\Omega |u(x)|^p dx\right)^{1/p} \quad and \quad |u|_\infty = esssup_{x\in\Omega}|u(x)|,$$

which makes $L^p(\Omega)$ and $L^\infty(\Omega)$ Banach spaces, respectively.
Let $W^{1,p}(\Omega)$ be the usual Sobolev space, i.e., $W^{1,p}(\Omega) = \left\{u \in L^p(\Omega) : \nabla u \in [L^p(\Omega)]^N\right\}$ which is endowed with norm

$$\|u\| = \left(\int_\Omega \left(|\nabla u(x)|^p + |u(x)|^p\right) dx\right)^{1/p}. \tag{1.3}$$

Then $W^{1,p}(\Omega)$ is a Banach space. $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. In $W_0^{1,p}(\Omega)$ we use the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p}. \tag{1.4}$$

Thanks to the Poincaré inequality, it is not difficult to see that (1.3) and (1.4) are equivalent. Therefore, in the sequel the norm in $W_0^{1,p}(\Omega)$ will be denoted by $\|\cdot\|$.
Let Ω be an open and bounded subset of $\mathbb{R}^N$, with $N \geq 3$. Then $W_0^{1,p}(\Omega)$ is embedded continuously in $L^q(\Omega)$, denoted by $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, for every $q \in [1, p^*]$, where $p^* = Np/N-p$. The embedding is compact if and only if $q \in [1, p^*)$ (see [5]).

It is said that $u \in W_0^{1,p}(\Omega)$ is a weak solution of $(\mathbf{1.1})$-$(\mathbf{1.2})$ if any $\phi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \left(\left(|\nabla u|^{p-2}\nabla u, \nabla\phi\right) + |u|^{p-2}u\phi\right) dx - \int_\Omega f(x,u)\,\phi dx = 0, \tag{1.5}$$

where "$(\cdot,\cdot)$" is the standard inner product in $\mathbb{R}^N$.
The energy functional corresponding to problem $(\mathbf{1.1})$-$(\mathbf{1.2})$ is defined as $J : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$J(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right) dx - \int_\Omega F(x,u)\,dx,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. By using the condition $(\mathbf{f}_1)$(see below) together with standard arguments, one can easily shows that J is well-defined on $W_0^{1,p}(\Omega)$ and is of class $C^1\left(W_0^{1,p}(\Omega), \mathbb{R}\right)$. Moreover, the derivative of J is the mapping $J' : W_0^{1,p}(\Omega) \to \left(W_0^{1,p}(\Omega)\right)^*$ given by the formula

$$\left\langle J'(u), v\right\rangle = \int_\Omega \left(\left(|\nabla u|^{p-2}\nabla u, \nabla v\right) + |u|^{p-2} uv\right) dx - \int_\Omega f(x,u)\,v dx,$$

for any $u, v \in W_0^{1,p}(\Omega)$. From the variational setting of problem $(\mathbf{1.1})$-$(\mathbf{1.2})$, i.e. (1.5), and definition of the derivative of J, it is obvious that weak solutions of $(\mathbf{1.1})$-$(\mathbf{1.2})$ correspond to critical points of J. Let λ_1 be the first eigenvalue of $-\triangle_p$ on $W_0^{1,p}(\Omega)$, that is,

$$\lambda_1 = \inf_{0\neq u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx},$$

it is well known that $\lambda_1 > 0$ [6].

2 Variational Approach

In this section, we give the first result of the present paper, based on the variational approach.

Proposition 2.1. *Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

$$\limsup_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-1}} < \lambda_1 \text{ uniformly a.e. } x \in \Omega. \qquad (\mathbf{f}_1)$$

Then problem $(\mathbf{1.1})$-$(\mathbf{1.2})$ has at least one solution. If in addition f also satisfies

$$\liminf_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} > \lambda_1 \text{ uniformly a.e. } x \in \Omega, \qquad (\mathbf{f}_2)$$

then problem $(\mathbf{1.1})$-$(\mathbf{1.2})$ has at least one nontrivial solution.

To obtain the result of Proposition 2.1, we need the following Proposition.

Proposition 2.2. (i) *The functional J is coercive.*
(ii) *The functional J is weakly lower semicontinuous.*

Proof. (i) From $(\mathbf{f}_1)$, there exists $a > 0$ and $b \in (0, \lambda_1)$ such that

$$|f(x,t)| \leq a + b\,|t|^{p-1} \quad \forall t \in \mathbb{R}.$$

Integrating, we get

$$|F(x,t)| \leq a\,|t| + \frac{b}{p}\,|t|^p \quad \forall t \in \mathbb{R}. \qquad (2.1)$$

Then, using the definition of the first eigenvalue and Poincaré inequality, we have

$$\left|\int_\Omega F(x,u)dx\right| \leq a\int_\Omega |u|\,dx + \frac{b}{p}\int_\Omega |u|^p\,dx \leq c\,\|u\| + \frac{b}{p\lambda_1}\,\|u\|^p\,.$$

Therefore,

$$\begin{aligned} J(u) &= \frac{1}{p}\|u\|^p - \int_\Omega F(u)\,dx \geq \frac{1}{p}\|u\|^p - c\|u\| - \frac{b}{p\lambda_1}\|u\|^p \\ &\geq \frac{1}{p}\left(1-\frac{b}{\lambda_1}\right)\|u\|^p - c\|u\|. \end{aligned}$$

Since $\lambda_1 > b$, J is coercive.

(ii) Let $\{u_k\} \subset W_0^{1,p}(\Omega)$ be a minimizing sequence for J. Since J is coercive, $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists $u \in W_0^{1,p}(\Omega)$ such that passing to a subsequence, still denoted by $\{u_n\}$, we have

$u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$;

$u_k \to u$ in $L^p(\Omega)$;

$u_k(x) \to u(x)$ a.e.in Ω;

and there exists $w \in L^p(\Omega)$ such that $|u_k(x)| \leq w(x)$ a.e. in Ω and for all $k \in \mathbb{N}$.

Since F is continuous, we have $F(x, u_k(x)) \to F(x, u(x))$ a.e. in Ω. Moreover, from (2.1), we also have

$$|F(x, u_k(x))| \leq a|u_k(x)| + \frac{b}{p}|u_k(x)|^p \leq c(|w(x)| + |w(x)|^p) \in L^1(\Omega)$$

a.e. in Ω and for all $k \in \mathbb{N}$. Consequently, by the dominated convergence theorem, we obtain that

$$\int_\Omega F(x, u_k)dx \to \int_\Omega F(x, u)dx. \tag{2.2}$$

On the other hand, since the norm $\|\cdot\|$ is weakly lower semicontinuous we have

$$\|u\|^p \leq \liminf_k \|u_k\|^p. \tag{2.3}$$

Thus, considering the relations (2.2) and (2.3) , we get

$$\begin{aligned} J(u) &= \frac{1}{p}\|u\|^p - \int_\Omega F(u)\,dx \\ &\leq \liminf_k \frac{1}{p}\|u_k\|^p - \lim_k \int_\Omega F(x, u_k)dx \\ &\leq \liminf_k \left(\frac{1}{p}\|u_k\|^p - \int_\Omega F(x, u_k)dx\right) = \liminf_k J(u_k). \end{aligned}$$

Therefore J is weakly lower semicontinuous. □

Proof of Proposition 2.1. From Proposition 2.2, we know that the functional J is coercive and weakly lower semicontinuous. Therefore, it has a global minimum u on $W_0^{1,p}(\Omega)$, which is a critical point [7]. We now show that under condition $(\mathbf{f}_2)$, u is not identically zero. From $(\mathbf{f}_2)$, there exists $\mu > \lambda_1$ and $\delta > 0$ such that

$$f(x,t) \geq \mu t^{p-1} \quad \forall t \in [0, \delta],$$

and hence

$$F(x,t) \geq \frac{\mu}{p}t^p \quad \forall t \in [0, \delta].$$

Let φ_1 be the first eigenfunction corresponding to λ_1 . Since $\varphi_1 \in L^\infty(\Omega)$, there exists $t > 0$ sufficiently small such that $t\varphi_1(x) < \delta$ for a.e. $x \in \Omega$. Therefore,

$$\begin{aligned} J(t\varphi_1) &= \frac{1}{p}\|t\varphi_1\|^p - \int_\Omega F(t\varphi_1)\,dx \leq \frac{t^p}{p}\|\varphi_1\|^p - \frac{\mu t^p}{p}\int_\Omega |\varphi_1|^p\,dx \\ &= \frac{t^p\lambda_1}{p}\int_\Omega |\varphi_1|^p\,dx - \frac{t^p\mu}{p}\int_\Omega |\varphi_1|^p\,dx = \frac{t^p}{p}(\lambda_1 - \mu)\int_\Omega |\varphi_1|^p\,dx < 0. \end{aligned}$$

Further, let u be the solution that minimizes J. Then

$$J(u) = \min_{v \in W_0^{1,p}(\Omega)} J(v) \leq J(t\varphi_1) < 0.$$

Since $J(0) = 0$, it follows that $u \neq 0$. This completes the proof. □

3 Monotone Operator Method

In this section, we use another method, named monotone operator method, for problem $(\mathbf{1.1})$-$(\mathbf{1.2})$. Therefore, we recall the following theorem, which is the key tool to get our second result.

Proposition 3.1 (**Browder**). *Let X be a reflexive real Banach space. Moreover, let $T : X \to X^*$ be an operator satisfying the conditions:*

(i) T *is bounded;*

(ii) T *is demicontinuous;*

(iii) T *is coercive;*

(iv) T *is monotone on the space X , i.e., for all $u, v \in X$ we have*

$$\langle T(u) - T(v), u - v \rangle \geq 0. \tag{3.1}$$

Then the equation

$$T(u) = h^* \tag{3.2}$$

has at least one solution $u \in X$ for every $h^* \in X^*$. If, moreover, the inequality (3.1) is strict for all $u, v \in X$, $u \neq v$, then the equation (3.2) has precisely one solution $u \in X$ for every $h^* \in X^*$.

We now proceed for the next result of the present paper. For this, we will establish the operator equations corresponding to problem $(\mathbf{1.1})$-$(\mathbf{1.2})$. Let us define the operators $J, F : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ by

$$\langle J(u), v \rangle = \int_\Omega \left(|\nabla u|^{p-2} (\nabla u, \nabla v) + |u|^{p-2} uv \right) dx \ \forall u, v \in W_0^{1,p}(\Omega),$$

$$\langle F(u), v \rangle = \int_\Omega f(x,u) v dx \ \forall u, v \in W_0^{1,p}(\Omega),$$

and set

$$T := J - F.$$

Then, from the monotone operator theory, the solution function $u \in W_0^{1,p}(\Omega)$ of $(\mathbf{1.1})$-$(\mathbf{1.2})$ satisfying the operator equation

$$T(u) := J(u) - F(u) = 0 \tag{3.3}$$

is also the solution of the integral equation

$$\int_\Omega \left(|\nabla u|^{p-2} (\nabla u, \nabla v) + |u|^{p-2} uv \right) dx - \int_\Omega f(x,u) v dx = 0 \ \forall v \in W_0^{1,p}(\Omega). \tag{3.4}$$

Namely, the existence of weak solution of problem $(\mathbf{1.1})$-$(\mathbf{1.2})$ is equivalent to the existence of solution of the operator equation (3.3) (see [8, 9])

Proposition 3.2. *Assume that the following assertions holds:*

$(\mathbf{f}_3)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function, and assume that there exist* $c_1, c_2 > 0$ *such that*

$$|f(x,t)| \leq c_1 + c_2 |t|^{q-1},$$

for all $x \in \Omega$ *and all* $t \in \mathbb{R}$*, where* $1 < q < p^*$*, with* $p > q$*;*

$(\mathbf{f}_4)$ $f(x,0) = 0$ *and* $(f(x,t) - f(x,s))(t-s) \leq 0$ *for all* $s, t \in \mathbb{R}$ *and a.e.* $x \in \Omega$*.*

Then problem $(\mathbf{1.1})$-$(\mathbf{1.2})$ *has an unique solution.*

Proof. It is obvious from $(\mathbf{f}_3)$ that T is well defined, bounded and continuous (and hence demicontinuous). From $(\mathbf{f}_4)$, for sufficiently large $\|u\|$, we have

$$\langle T(u), u\rangle = \int_\Omega (|\nabla u|^p + |u|^p)\, dx - \int_\Omega f(x,u)u dx \geq \|u\|^p.$$

This shows that T is coercive. Let us show the monotonicity of T. If $u = v$, the assertion of theorem is obvious. For the case $u \neq v$, from $(\mathbf{f}_4)$, it reads

$$\begin{aligned} \langle T(u) - T(v), u - v\rangle &\geq \int_\Omega \left(|\nabla u|^{p-2} (\nabla u, \nabla u - \nabla v) + |u|^{p-2} u (u - v)\right) dx \\ &\quad - \int_\Omega \left(|\nabla v|^{p-2} (\nabla v, \nabla u - \nabla v) + |v|^{p-2} v (u - v)\right) dx \\ &= \int_\Omega \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla u - \nabla v\right) dx \\ &\quad + \int_\Omega \left(|u|^{p-2} u - |v|^{p-2} v\right)(u - v)\, dx. \end{aligned}$$

Now, we apply the following well-known vectorial inequality: For all $\xi, \eta \in \mathbb{R}^N$, it holds [10],

$$\begin{aligned} \left(|\xi|^{r-2} \xi - |\eta|^{r-2} \eta, \xi - \eta\right) &\geq 2^{1-r}\ |\xi - \eta|^r\ ,\ r \geq 2, \\ \left(|\xi|^{r-2} \xi - |\eta|^{r-2} \eta, \xi - \eta\right) &\geq (r-1)\ \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-r}}\ ,\ 1 < r < 2, \end{aligned}$$

Then, we get

$$\langle T(u) - T(v), u - v\rangle \geq 2^{1-p} \int_\Omega \left(|\nabla u - \nabla v|^p + |u - v|^p\right) dx > 0,\ p \geq 2, \tag{3.5}$$

$$\langle T(u) - T(v), u - v\rangle \geq (p-1) \int_\Omega \left(\frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}} + \frac{|u - v|^2}{(|u| + |v|)^{2-p}}\right) dx > 0, 1 < p < 2, \tag{3.6}$$

This implies the monotonicity of T. As a consequence of Proposition 3.1, the equation

$$T(u) = J(u) - F(u) = h^*$$

has at least one solution $u \in W_0^{1,p}(\Omega)$ for every $h^* \in (W_0^{1,p}(\Omega))^*$. Moreover , since inequalities $(3.5), (3.6)$ are strict, it follows then from Proposition 3.1 that there is a unique solution of (3.3), which in turn is a unique weak solution of $(\mathbf{1.1})$-$(\mathbf{1.2})$. □

4 Conclusion

In the present paper, the existence results of a quasilinear elliptic equation is investigated. For this, the problem is settled in two different manner: variational approach and monotone operator method. The problem is studied in classical Sobolev space. If the results could be extended to the variable exponent Sobolev spaces, it would be more interesting. But when one intends to do so, the main difficulty would be the characterization of the first eigenvalue.

Competing Interests

The author declares that no competing interests exist.

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