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Periodic Solutions of a Time Delay Stage-structured Prey-predator Model

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Abstract

The dynamics of a time delay stage-structured prey-predator model is investigated. Firstly, through the analysis of the eigenvalues, the effect of time delay on the stability of the positive equilibrium and the existence of Hopf bifurcation are obtained. Further, the stability and the direction of Hopf bifurcation periodic solution near the first critical value are given utilizing the normal form method and the center manifold theorem.

Keywords: Stability; time delay; Hopf bifurcation; stage structure.
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1 Introduction

The prey-predator model is a mathematical model to study the relationship between the predator and the prey, which has attracted many scholars [1-4]. In general, we assume the individual ability is the same in the population of prey and predator, but this assumption is often not in accord with the fact. Because in nature, biological individual growth needs to go through a plurality of stages, such

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as juvenile, adult, the elderly, and the abilities of survival, predation and reproduction at each stage are not the same, it is necessary to research predator-prey model with stage structure. The effect of stage structure on the prey-predator model has been widely concerned [5-8].

[5] established the following mathematical model of two species with stage structure and proved that the globally asymptotical stability of the positive equilibrium:

$$\begin{cases} \dot{x}_J(t) = r_1 x_A - d_1 x_J - \alpha x_J - s_1 x_J^2 - \beta x_J y, \\ \dot{x}_A(t) = \alpha x_J - d_2 x_A, \\ \dot{y}(t) = \beta x_J y - d_3 y - s_2 y^2. \end{cases}$$

Because the reproduction of predator after predating the prey is not instantaneous, the time delay τ can be introduced to denote this time. Moreover, previous researches also show that time delay has a significant impact on stage structured prey-predator system (see [9-13]). In this paper, the model with delay below is investigated:

$$\begin{cases} \dot{x}_J(t) = r_1 x_A - d_1 x_J - \alpha x_J - s_1 x_J^2 - \beta x_J y, \\ \dot{x}_A(t) = \alpha x_J - d_2 x_A, \\ \dot{y}(t) = \beta x_J(t-\tau) y(t-\tau) - d_3 y - s_2 y^2 \end{cases} \tag{1.1}$$

subject to the following initial conditions:

$$\begin{aligned} x_J(\theta) &= \phi_1(\theta) > 0, \;\; \theta \in [-\tau, 0), \;\; \phi_1(0) > 0, \\ x_A(0) &= \phi_2(0) > 0, \\ y(\theta) &= \phi_3(\theta) > 0, \;\; \theta \in [-\tau, 0), \;\; \phi_3(0) > 0. \end{aligned}$$

Some explanations about the model are as follows:
(i) All these population are growing in a closed homogeneous environment.
(ii) At any time $(t > 0)$, the birth of the juveniles are proportional to the existing adult population with proportionality constant r_1; the rate of transformation of the adults is proportional to the existing juveniles with proportionality constant α.
(iii) The death rates of the juveniles, adults and the predators are proportional to that existing juvenile, adult and predator population with respective proportionality constants $d_1,\ d_2$ and d_3.
(iv) The juveniles are density restricted and the predators compete among themselves for food and hence the terms $-s_1 x_J^2$ and $-s_2 y^2$ in (1.1). s_1 and s_2 are the intra-specific competition coefficients of the prey (juvenile) and predator population respectively.
(v) The predator consumes the prey (juvenile) at the rate β, which explains the term $-\beta x_J y$ in the first equation.
(vi) All the constants, namely, $r_1,\ d_1,\ \alpha,\ s_1,\ \beta,\ d_2,\ d_3$ and s_2 are positive.

2 Local Stability and Hopf Bifurcation

For the boundedness and positivity of the solutions for the system (1.1), [14] has obtained the corresponding conditions. So we don't repeat it. First, we simplify the system (1.1). Let $y_1 = \frac{\beta}{d_2} x_J,\ \ y_2 = \frac{\beta}{\alpha} x_A,\ \ y_3 = \frac{s_2}{d_2} y,\ \ \mathrm{d}t = \frac{1}{d_2}\mathrm{d}s$ and still use t to indicate the time, then we have

$$\begin{cases} \dot{y}_1(t) = a y_2 - b y_1 - c y_1^2 - d y_1 y_3, \\ \dot{y}_2(t) = y_1 - y_2, \\ \dot{y}_3(t) = y_1(t-\tau) y_3(t-\tau) - e y_3 - y_3^2 \end{cases} \tag{2.1}$$

where $\dot{y}_i = \mathrm{d}y_i/\mathrm{d}t,\ a = \alpha r_1/d_2^2,\ b = (d_1+\alpha)/d_2,\ c = s_1/\beta,\ d = \beta/s_2,\ e = d_3/d_2.$

The system (2.1) has equilibria

$$E_0(0,0,0),\ E_1\left(\frac{a-b}{c},\frac{a-b}{c},0\right),\ E_2(\bar{y}_1,\bar{y}_2,\bar{y}_3)$$

where

$$\bar{y}_1=\frac{a-b+de}{c+d},\ \ \bar{y}_2=\frac{a-b+de}{c+d},\ \ \bar{y}_3=\frac{a-b-ce}{c+d}.$$

When $a-b>ce$, E_2 is a positive equilibrium. The characteristic equation of its corresponding linear system around E_2 is

$$\begin{aligned}&\lambda^3+[b+e+1+2c\bar{y}_1+(d+2)\bar{y}_3]\lambda^2+[b+2c\bar{y}_1+d\bar{y}_3+(b+2c\bar{y}_1+d\bar{y}_3)\\&(e+2\bar{y}_3)+e+2\bar{y}_3-a]\lambda+(b-a+2c\bar{y}_1+d\bar{y}_3)(e+2\bar{y}_3)\\&-\bar{y}_1e^{-\lambda\tau}[\lambda^2+(b+1+2c\bar{y}_1)\lambda+b-a+2c\bar{y}_1]=0.\end{aligned}\tag{2.2}$$

Refer to the eigenvalue analysis in [15] and when $\tau=0$, the characteristic equation becomes

$$\lambda^3+(a+1+c\bar{y}_1+\bar{y}_3)\lambda^2+(c\bar{y}_1+(a+1+(c+d)\bar{y}_1)\bar{y}_3)\lambda+(c+d)\bar{y}_1\bar{y}_3=0.$$

Because $a+1+c\bar{y}_1+\bar{y}_3>0,\ (c+d)\bar{y}_1\bar{y}_3>0,\ (a+1+c\bar{y}_1+\bar{y}_3)(c\bar{y}_1+(a+1+(c+d)\bar{y}_1)\bar{y}_3)-(c+d)\bar{y}_1\bar{y}_3>0$, according to Routh-Hurwitz theorem, all the eigenvalues have negative real parts.

When $\tau\neq 0$, let $\lambda=\pm\mathrm{i}\omega\ (\omega>0)$ be roots of Eq. (2.2), then ω satisfies

$$\omega^6+(p_1^2-2p_2-q_1^2)\omega^4+(p_2^2-2p_1p_3+2q_1q_3-q_2^2)\omega^2+(p_3-q_3)(p_3+q_3)=0\tag{2.3}$$

where

$$\begin{aligned}p_1&=b+e+1+2c\bar{y}_1+(d+2)\bar{y}_3,\\p_2&=b+2c\bar{y}_1+d\bar{y}_3+(b+2c\bar{y}_1+d\bar{y}_3)(e+2\bar{y}_3)+e+2\bar{y}_3-a,\\p_3&=(b-a+2c\bar{y}_1+d\bar{y}_3)(e+2\bar{y}_3),\\q_1&=-\bar{y}_1,\\q_2&=-\bar{y}_1(b+1+2c\bar{y}_1),\\q_3&=-\bar{y}_1(b-a+2c\bar{y}_1).\end{aligned}$$

It is easy to prove that

$$p_1^2-2p_2-q_1^2=(4+d^2)\bar{y}_3^2+2(2e+bd)\bar{y}_3+4bc\bar{y}_1+4cd\bar{y}_1\bar{y}_3+e^2+b^2+2a+1>0$$

and

$$p_3-q_3=(e+2\bar{y}_3)d\bar{y}_3+(e+2\bar{y}_3+\bar{y}_1)(b+2c\bar{y}_1-a)>(e+2\bar{y}_3)d\bar{y}_3+(e+2\bar{y}_3+\bar{y}_1)(ce+2c\bar{y}_1)>0,$$

so Eq. (2.3) has a positive root ω_0 when $p_3+q_3<0$. Then

$$\tau_j=\frac{1}{\omega_0}[\arccos\frac{(q_1\omega^2-q_3)(p_1\omega^2-p_3)+q_2\omega^2(p_2-\omega^2)}{\omega^2q_2^2+(q_1\omega^2-q_3)^2}+2j\pi],\ \ j=0,1,2,\cdots.$$

Let $\lambda=\alpha(\tau)+\mathrm{i}\omega(\tau)$ be the root of Eq. (2.2) satisfying $\alpha(\tau_j)=0$ and $\omega(\tau_j)=\omega_0$. [14] has proved

$$\left.\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right|_{\tau=\tau_j}>0,\ \ j=0,1,2,\cdots.$$

To sum up, we can get the stability of the positive equilibrium E_2 and the existence of Hopf bifurcation.

Theorem 2.1 Suppose $a-b>ce$ and $p_3+q_3<0$.
(i) E_2 is asymptotically stable for $\tau\in[0,\tau_0)$ and unstable for $\tau\in(\tau_0,+\infty)$;
(ii) System (1.1) undergoes Hopf bifurcation at the positive equilibrium E_2 when $\tau=\tau_j,\ j=0,1,2,\cdots$.

3 Direction and Stability of Hopf Bifurcation

We first rescale the time by $t \mapsto t/\tau$ to normalize the delay and translate E_2 to zero equilibrium so that system (2.1) can be written as the form

$$\begin{cases} \dot{y}_1(t) = \tau[a(y_2 + \bar{y}_2) - b(y_1 + \bar{y}_1) - c(y_1 + \bar{y}_1)^2 - d(y_1 + \bar{y}_1)(y_3 + \bar{y}_3)], \\ \dot{y}_2(t) = \tau[(y_1 + \bar{y}_1) - (y_2 + \bar{y}_2)], \\ \dot{y}_3(t) = \tau[(y_1(t-1) + \bar{y}_1)(y_3(t-1) + \bar{y}_3) - e(y_3 + \bar{y}_3) - (y_3 + \bar{y}_3)^2]. \end{cases} \tag{3.1}$$

Let $\tau = \tau_0 + \mu,\ \mu \in \mathbb{R}$. Then $\mu = 0$ is the critical value of Hopf bifurcation for Eq. (3.1).

Notating $C = C([-1,\ 0],\ \mathbb{R}^3)$, by Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, \mu)$ $(-1 \leq \theta \leq 0)$ such that

$$L_\mu(\varphi) = \int_{-1}^{0} \mathrm{d}\eta(\theta, \mu)\varphi(\theta)$$

where

$$A = \begin{pmatrix} -b - 2c\bar{y}_1 - d\bar{y}_3 & a & -d\bar{y}_1 \\ 1 & -1 & 0 \\ 0 & 0 & -e - 2\bar{y}_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{y}_3 & 0 & \bar{y}_1 \end{pmatrix},$$

$$\eta(\theta, \mu) = \begin{cases} (\tau_0 + \mu)A, & \theta = 0, \\ 0, & \theta \in (-1,\ 0), \\ -(\tau_0 + \mu)B, & \theta = -1. \end{cases}$$

Denote

$$\begin{aligned} F(\mu, \varphi) =& (\tau_0 + \mu) \begin{pmatrix} -c\bar{y}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\bar{y}_3 \end{pmatrix} \varphi^2(0) + (\tau_0 + \mu) \begin{pmatrix} 0 & -d\bar{y}_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(0)\varphi_2(0) \\ \varphi_1(0)\varphi_3(0) \\ \varphi_2(0)\varphi_3(0) \end{pmatrix} \\ &+ (\tau_0 + \mu) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\bar{y}_3}{2} & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(-1)\varphi_2(-1) \\ \varphi_1(-1)\varphi_3(-1) \\ \varphi_2(-1)\varphi_3(-1) \end{pmatrix}. \end{aligned}$$

For $\varphi \in C^1([-1,\ 0],\ \mathbb{R}^3)$, define

$$A(\mu)\varphi = \begin{cases} \mathrm{d}\varphi(\theta)/\mathrm{d}\theta, & \theta \in [-1,\ 0), \\ \int_{-1}^{0} d\eta(t, \mu)\varphi(t), & \theta = 0, \end{cases}$$

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1,\ 0), \\ F(\mu, \varphi), & \theta = 0. \end{cases}$$

Eq. (3.1) can be written as

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{3.2}$$

where $u = (y_1,\ y_2,\ y_3)^T$ and $u_t = u(t + \theta),\ \theta \in [-1,\ 0]$.

For $\psi \in C^1([0,\ 1],\ \mathbb{R}^3)$, define

$$A^*\psi(s) = \begin{cases} \mathrm{d}\psi(s)/\mathrm{d}s, & s \in (0,\ 1], \\ \int_{-1}^{0} d\eta(t, 0)\psi(-t), & s = 0. \end{cases}$$

For $\varphi \in C^1([-1,\ 0],\ \mathbb{R}^3)$ and $\psi \in C^1([0,\ 1],\ \mathbb{R}^3)$, define the bilinear form

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) \varphi(\xi) \mathrm{d}\xi$$

where $\eta(\theta) = \eta(\theta, 0)$. Then A^* and $A(0)$ are adjoint operators, and $\pm\mathrm{i}\tau_0\omega_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*.

Let $q(\theta), q^*(s)$ satisfying $\langle q^*, q\rangle = 1$ and $\langle q^*, \bar{q}\rangle = 0$ be eigenvectors of $A,\ A^*$ corresponding to $\mathrm{i}\tau_0\omega_0$ and $-\mathrm{i}\tau_0\omega_0$, respectively. By direct computation, we obtain that

$$q(\theta) = \begin{pmatrix} d\bar{y}_1 \\ d\bar{y}_1(1+\mathrm{i}\omega_0) \\ a-(b+2c\bar{y}_1+d\bar{y}_3+\mathrm{i}\omega_0)(1+\mathrm{i}\omega_0) \end{pmatrix} e^{\mathrm{i}\tau_0\omega_0\theta},$$

$$q^*(s) = D(a\bar{y}_3,\ \bar{y}_3(1-\mathrm{i}\omega_0),\ (b+2c\bar{y}_1+d\bar{y}_3-\mathrm{i}\omega_0)(1-\mathrm{i}\omega_0)-a)e^{\mathrm{i}\tau_0\omega_0 s}$$

where

$$\begin{aligned} D =& \{d\bar{y}_1\bar{y}_3[\tau_0((b+2c\bar{y}_1+d\bar{y}_3-\mathrm{i}\omega_0)(1-\mathrm{i}\omega_0)-a)e^{\mathrm{i}\omega_0\tau_0}+a+(1-\mathrm{i}\omega_0)^2] \\ &-((b+2c\bar{y}_1+d\bar{y}_3-\mathrm{i}\omega_0)(1-\mathrm{i}\omega_0)-a)^2(1+\tau_0\bar{y}_1e^{\mathrm{i}\omega_0\tau_0})\}^{-1}. \end{aligned}$$

Using the same notation as in [16], define

$$z(t) = \langle q^*, u_t\rangle, w(t,\theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}.$$

On the center manifold $\mathcal{C}_0$, we have

$$w(t,\theta) = w(z(t), \bar{z}(t), \theta),$$

where

$$w(z(t), \bar{z}(t), \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

z and $\bar{z}$ are local coordinates for the center manifold $\mathcal{C}_0$ in the direction of q^* and $\bar{q^*}$. Notice that w is real if u_t is real. We only consider real solutions. Since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= \mathrm{i}\tau_0\omega_0 z + \langle q^*(s), f(0, w+2\mathrm{Re}\{z(t)q(\theta)\}\rangle \\ &= \mathrm{i}\tau_0\omega_0 z + \bar{q^*}(0)f(0, w(z,\bar{z},0)+2\mathrm{Re}\{z(t)q(0)\}) \\ &\stackrel{\text{def}}{=} \mathrm{i}\tau_0\omega_0 z(t) + \bar{q^*}(0)f_0(z,\bar{z}) \end{aligned} \tag{3.3}$$

for solution $u_t \in \mathcal{C}_0$. We rewrite this as

$$\dot{z}(t) = \mathrm{i}\tau_0\omega_0 z(t) + g(z,\bar{z}), \tag{3.4}$$

where

$$f_0(z,\bar{z}) = f_{z^2}\frac{z^2}{2} + f_{\bar{z}^2}\frac{\bar{z}^2}{2} + f_{z\bar{z}}z\bar{z} + f_{z^2\bar{z}}\frac{z^2\bar{z}}{2} + \cdots, \tag{3.5}$$

$$\begin{aligned} g(z,\bar{z}) =& \bar{q^*}(0)f(0, w(z,\bar{z},0)+2\mathrm{Re}\{z(t)q(0)\}) \\ =& g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \end{aligned} \tag{3.6}$$

Comparing the coefficients of Eq. (3.3) and Eq. (3.4), noticing Eq. (3.6), we have

$$\begin{aligned} g_{20} &= \bar{q^*}(0)f_{z^2}, \\ g_{11} &= \bar{q^*}(0)f_{z\bar{z}}, \\ g_{02} &= \bar{q^*}(0)f_{\bar{z}^2}, \\ g_{21} &= \bar{q^*}(0)f_{z^2\bar{z}}. \end{aligned}$$

By

$$w(z,\bar{z},\theta) = u_t(\theta) - 2\mathrm{Re}(z(t)q(\theta)) = u_t - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta),$$

then

$$\begin{aligned} u_t(\theta) = &\begin{pmatrix} d\bar{y}_1 \\ d\bar{y}_1(1+\mathrm{i}\omega_0) \\ a-(b+2c\bar{y}_1+d\bar{y}_3+\mathrm{i}\omega_0)(1+\mathrm{i}\omega_0) \end{pmatrix} e^{\mathrm{i}\tau_0\omega_0\theta} z(t) \\ &+ \begin{pmatrix} d\bar{y}_1 \\ d\bar{y}_1(1-\mathrm{i}\omega_0) \\ a-(b+2c\bar{y}_1+d\bar{y}_3-\mathrm{i}\omega)(1-\mathrm{i}\omega_0) \end{pmatrix} e^{-\mathrm{i}\tau_0\omega_0\theta} \bar{z}(t) \\ &+ w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots , \end{aligned}$$

where $w_{ij}(\theta) = (w_{ij}^1(\theta), w_{ij}^2(\theta), w_{ij}^3(\theta))^T$.

By

$$\begin{aligned} f_0 = &\tau_0 \begin{pmatrix} -c\bar{y}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\bar{y}_3 \end{pmatrix} \varphi^2(0) + \tau_0 \begin{pmatrix} 0 & -d\bar{y}_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(0)\varphi_2(0) \\ \varphi_1(0)\varphi_3(0) \\ \varphi_2(0)\varphi_3(0) \end{pmatrix} \\ &+ \tau_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\bar{y}_3}{2} & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(-1)\varphi_2(-1) \\ \varphi_1(-1)\varphi_3(-1) \\ \varphi_2(-1)\varphi_3(-1) \end{pmatrix}, \end{aligned}$$

we have

$$\begin{aligned} g_{20} = &-2\tau_0\bar{D}[\bar{q_1^*}(0)(c\bar{y}_1 q_1^2(0) + d\bar{y}_3 q_1(0)q_3(0)) + \bar{q_3^*}(0)(\bar{y}_3 q_3^2(0) \\ &- \frac{1}{2}\bar{y}_3 q_1(-1)q_3(-1))], \\ g_{11} = &-\tau_0\bar{D}[\bar{q_1^*}(0)(2c\bar{y}_1 q_1(0)\bar{q}_1(0) + d\bar{y}_3 q_1(0)\bar{q}_3(0) + d\bar{y}_3 q_3(0)\bar{q}_1(0)) + \bar{q_3^*}(0) \\ &(2\bar{y}_3 q_3(0)\bar{q}_3(0) - \frac{1}{2}\bar{y}_3 q_1(-1)\bar{q}_3(-1) - \frac{1}{2}\bar{y}_3 q_3(-1)\bar{q}_1(-1))], \\ g_{02} = &-2\tau_0\bar{D}[\bar{q_1^*}(0)(c\bar{y}_1\bar{q}_1^2(0) + d\bar{y}_3\bar{q}_1(0)\bar{q}_3(0)) + \bar{q_3^*}(0)(\bar{y}_3\bar{q}_3^2(0) \\ &- \frac{1}{2}\bar{y}_3\bar{q}_1(-1)\bar{q}_3(-1))], \\ g_{21} = &-2\tau_0\bar{D}[\bar{q_1^*}(0)(c\bar{y}_1\bar{q}_1(0)w_{20}^1(0) + 2c\bar{y}_1 q_1(0)w_{11}^1(0) + d\bar{y}_3 q_1(0)w_{11}^3(0) \\ &+ \frac{d\bar{y}_3}{2}\bar{q}_1(0)w_{20}^3(0) + \frac{d\bar{y}_3}{2}\bar{q}_3(0)w_{20}^1(0) + d\bar{y}_3 q_3(0)w_{11}^1(0)) \\ &+ \bar{q_3^*}(0)(\bar{y}_3\bar{q}_3(0)w_{20}^3(0) + 2\bar{y}_3 q_3(0)w_{11}^3(0) + \frac{\bar{y}_3}{2}q_1(-1)w_{11}^3(-1) \\ &+ \frac{\bar{y}_3}{2}\bar{q}_1(-1)w_{20}^3(-1) + \frac{\bar{y}_3}{2}\bar{q}_3(-1)w_{20}^1(-1) + \bar{y}_3 q_3(-1)w_{11}^1(-1))] \end{aligned}$$

where

$$\begin{aligned} w_{20}(\theta) &= \frac{\mathrm{i}g_{20}q(0)e^{\mathrm{i}\tau_0\omega_0\theta}}{\tau_0\omega_0} + \frac{\mathrm{i}\bar{g}_{20}\bar{q}(0)e^{-\mathrm{i}\tau_0\omega_0\theta}}{3\tau_0\omega_0} + E_1 e^{2\mathrm{i}\tau_0\omega_0\theta}, \\ w_{11}(\theta) &= -\frac{\mathrm{i}g_{11}q(0)e^{\mathrm{i}\tau_0\omega_0\theta}}{\tau_0\omega_0} + \frac{\mathrm{i}\bar{g}_{11}\bar{q}(0)e^{-\mathrm{i}\tau_0\omega_0\theta}}{\tau_0\omega_0} + E_2. \end{aligned}$$

$E_1,\ E_2$ satisfy the following equations, respectively.

$$\begin{aligned} &[2\mathrm{i}\tau_0\omega_0\mathrm{Id} - \int_{-1}^{0} e^{2\mathrm{i}\tau_0\omega_0\theta}\mathrm{d}\eta(\theta)]E_1 = f_{z^2}, \\ &\int_{-1}^{0}\mathrm{d}\eta(\theta)E_2 = -f_{z\bar{z}}. \end{aligned}$$

Define

$$
\begin{aligned}
C_1(0) &= \frac{\mathrm{i}}{2\omega_0\tau_0}(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{\mathrm{Re}C_1(0)}{\mathrm{Re}(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}|_{\lambda=\mathrm{i}\omega_0,\tau=\tau_0})},\\
\beta_2 &= 2\mathrm{Re}C_1(0).
\end{aligned}
$$

Because $\mathrm{Re}(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}|_{\lambda=\mathrm{i}\omega_0,\tau=\tau_0}) > 0$, the following conclusion is right.

Theorem 3.1 Suppose $a - b > ce$ and $p_3 + q_3 < 0$. If $\beta_2 < 0$ (> 0), then bifurcating periodic solution of system (1.1) at E_2 when $\tau = \tau_0$ is orbitally asymptotically stable (unstable), and the direction of the bifurcation is supercritical (subcritical).

4 Numerical Simulations

Choosing $r_1 = 7,\ d_1 = 1,\ d_2 = 0.9,\ d_3 = 0.7,\ \alpha = 2,\ \beta = 1.5,\ s_1 = 0.2$ and $s_2 = 0.8$, system (1.1) can be expressed as follows:

$$
\begin{cases}
\dot{x}_J(t) = 7x_A - x_J - 2x_J - 0.2x_J^2 - 1.5x_J y,\\
\dot{x}_A(t) = 2x_J - 0.9x_A,\\
\dot{y}(t) = 1.5x_J(t-\tau)y(t-\tau) - 0.7y - 0.8y^2.
\end{cases}
\tag{4.1}
$$

Then $a - b - ce = 13.8469 > 0$ and $p_3 + q_3 = -44.0030 < 0$. The conditions are satisfied and $\tau_0 = 7.4421$.

By Theorem 2.1, the positive equilibrium $(7.6725,\ 7.6725,\ 6.8947)$ of system (4.1) is asymptotically stable when $\tau < 7.4421$ and unstable when $\tau > 7.4421$ (see Figure 1 at $\tau = 5$).

Using the formula of β_2, we can obtain $\beta_2 = -0.5254$ at $\tau = 7.4421$, respectively. By Theorem 3.1, system (4.1) has a orbitally asymptotically stable periodic solution near τ_0. Then the direction of the bifurcation at τ_0 is supercritical. As shown in Figure 2 at $\tau = 8$.

5 Conclusions

In this paper, we investigate the Hopf bifurcation of a time delay stage-structured prey-predator model. Using the normal form method for functional differential equations (FDEs) and the center manifold theory in [16], we have obtained the properties of Hopf bifurcation.

The previous results have shown that the positive equilibrium of system (1.1) is globally asymptotically stable when the time delay $\tau = 0$. Our discussions indicate that the existence of time delay can make the stability of equilibrium change, and the periodic solutions appear at some critical values. From the perspective of dynamics, the solutions of system have changed significantly and the system presents complex behavior. In addition, the results indicate the population of one species does not change infinitely as another species varies, and there are restrictions between them.

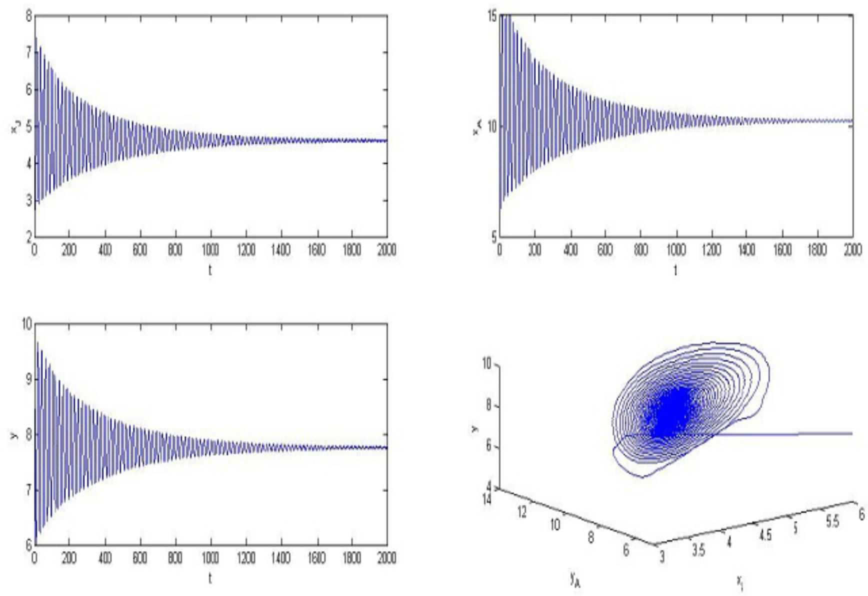

Figure 1. The positive equilibrium $(7.6725,\ 7.6725,\ 6.8947)$ of system (5.1) is asymptotically stable when $\tau = 5 < \tau_0 = 7.4421$.

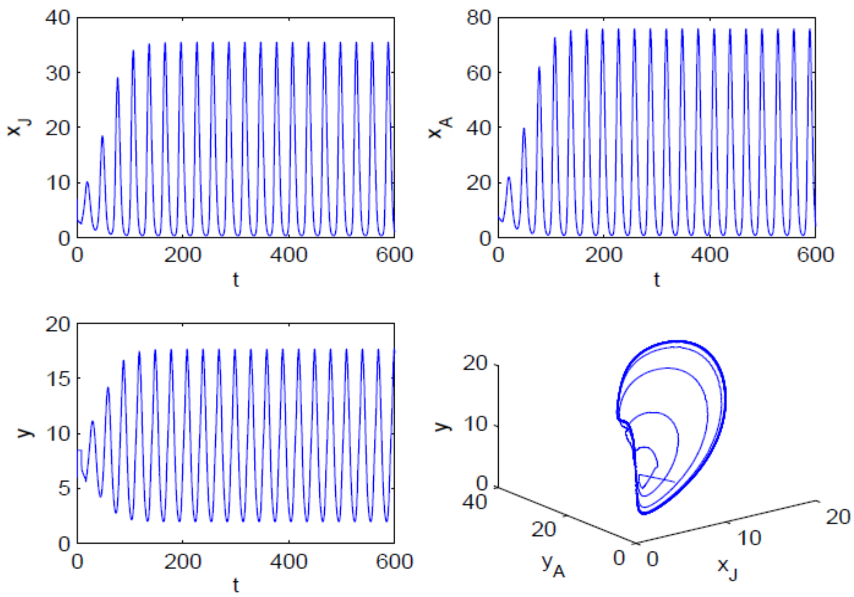

Figure 2. Numerical simulations of a periodic solution to system (5.1) when $\tau = 8 > \tau_0 = 7.4421$.

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Competing Interests

The authors declare that no competing interests exist.

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