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On Algebraic Properties of Fuzzy Membership Sequenced Multisets

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Abstract

Since the original paper by Yager, different definitions of the concept of fuzzy multisets (bags), and the corresponding operations are available in the literature, as well as some extensions of the union, intersection and difference operators of sets, and new algebraic operators. In this paper, a study of some algebraic properties of these operations including idempotency, identity, absorption, associativity, distributivity, demorgan's laws and the principle of Inclusion/Exclusion is presented in the context of membership sequence of fuzzy multisets. Also, the compatibility of the union and intersection on these structures and sets via their root sets is established.

Keywords: Multiset, fuzzy multiset, Membership sequence of fuzzy multiset, Algebraic properties, root set, principle of Inclusion/Exclusion.

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1 Introduction

Many problems in various disciplines such as engineering, medicine, economics, environmental sciences, and so forth have various uncertainties. These problems, which one comes face to face within life, cannot be solved using classical Mathematics method. In classical mathematics, a mathematical model of an object is devised and the notion of the exact solution of this model is determined. As for these problems of uncertainty, the mathematical model is too complex, the exact solution cannot be found. There are several well-known theories to describe uncertainty. Considering the uncertainty factor, Lotfi Zadeh [1] introduced Fuzzy sets, in which a membership function assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration.

Fuzzy set theory has become a very important tool to solve problems and provides an appropriate framework for representing vague concepts by allowing partial membership. The theory has been studied by both mathematicians and computer scientists and many applications of fuzzy set theory have arisen over the years, such as fuzzy control systems, fuzzy automata and fuzzy logic [2], analysis and fuzzy topology [3,4,5,6,7] etc..

However, many fields of modern mathematics have been emerged by violating a basic principle of a given theory only because useful structures could be defined this way. Set is a well-defined collection of distinct objects, that is, the elements of a set are pair wise different. If we relax this restriction and allow repeated occurrences of any element, then we can get a mathematical structure that is known as multisets or bags. For example, the prime factorization of an integer $n > 0$ is a multiset whose elements are primes. The number 120 has the prime factorization $120 = 2^3 3^1 5^1$ which gives the multiset $\{2,2,2,3,5\}$.

A complete account of the development of multiset theory and applications can be seen in Blizard [8,9,10] respectively.

As a generalization of multiset, Yager [11] introduced the concept of fuzzy multiset. An element of a fuzzy multiset can occur more than once with possibly the same or different membership values.

Fuzzy multisets have found several applications in areas including mathematics [12], statistics and engineering [13], computer science [14], medicine [15], economics and environmental sciences [16].

Since the original paper by Yager, different definitions of the concept of fuzzy multiset, and the corresponding operators, are available in the literature, as well as some extensions of the union, intersection and difference operators of sets, and new algebraic operators [15,17,18,19]. In this paper, a study of the algebraic properties of these operators in the context of membership sequence is presented: Some basic definitions and notations are given in section 2. The operations on fuzzy multisets with membership sequence are recalled and some results also proposed in section 3. Section 4 summarizes the entire study.

2 Preliminary, Definition and Notation

Definition 2.1 [20]: A collection of elements which may contain duplicates is called a multiset (mset for short). Formally, if X is a finite set of elements, a mset M drawn from the set X is represented by a function $Count\ M$ or C_M defined as $C_M: X \to \mathbb{N}$ where $\mathbb{N}$ represents the set of non-negative integers. For each $x \in X, C_M(x)$ is the characteristic value of x in M. Note that M, $C_M(x)$ are assumed finite and $C_M(x) = 0$ implies that $x \notin M$. For a mset, different expressions such as $M = \{m_1/x_1, m_2/x_2, \dots, m_n/x_n\}$

and
$M = \left\{\overbrace{x_1, x_1, \dots, x_1}^{m_1}, \overbrace{x_2, x_2, \dots, x_2}^{m_2}, \dots, \overbrace{x_n, x_n, \dots, x_n}^{m_n}\right\}$ are used indicating m_i times membership of x_i in M. In other words, $M = \{a, a, a, a, b, b, b, c, c, c\}$ can be rewritten $M = \{4/a, 3/b, 3/c\}$ where
$C_M(a) = 4, C_M(b) = 3, C_M(c) = 3$. A mset M is a set if $C_M(x) = 0$ or 1 for all $x \in X$. Note that M, $C_M(x)$ are assumed finite and $C_M(x) = 0$ implies that $x \notin M$. A mset is said to be empty or null denoted $\emptyset$ if $C_\emptyset(x) = 0$ for all $x \in X$. The set of all finite msets over a finite set $\mathcal{S}$ is denoted by $\mathfrak{M}(\mathcal{S})$.

Definition 2.2 [11]**:** For any msets
$M, N \in \mathfrak{M}(\mathcal{S})$, the following operations are defined:

(i). (inclusion) : $M \subseteq N \leftrightarrow C_M(x) \leq C_N(x)$ for all $x \in \mathcal{S}$
(ii). (equality): $M = N \leftrightarrow C_M(x) = C_N(x)$ for all $x \in \mathcal{S}$
(iii). (Union): $C_{M\cup N}(x) = max\{C_M(x), C_N(x)\}$ for all $x \in \mathcal{S}$
(iv). (intersection): $C_{M\cap N}(x) = min\{C_M(x), C_N(x)\}$ for all $x \in \mathcal{S}$
(v). (addition): $C_{M+N}(x) = C_M(x) + C_N(x)$ for all $x \in \mathcal{S}$

Theorem 2.1 [21]. For any msets
$M, N, P \in \mathfrak{M}(\mathcal{S})$, the following properties are established:
(i). $M + (N + P) = (M + N) + P$
(ii). $M + N = (M \cup N) + (M \cap N)$

Definition 2.3 . Let $\mathcal{S}$ be a finite set of elements. A fuzzy mset (fuzzy bag) A drawn from $\mathcal{S}$ is a function $\Psi_A: \mathcal{S} \to \mathfrak{M}([0,1])$.

Where $\mathfrak{M}([0,1])$ is the set of all finite msets drawn from the unit interval [0,1].

In particular, a fuzzy mset is a mset in which each occurrence of each element is associated with a grade of membership. For example if $\mathcal{S} = \{a, b, c, d\}$ and
$A = \{(a, 0.2), (a, 0.3), (b, 1), (b, 0.5), (b, 0.5)\}$, A can be rewritten:
$A = \{\{0.2,0.3\}/a, \{1,0.5,0.5\}/b\}$ where
$\Psi_A(a) = \{0.2,0.3\}$ and $\Psi_A(b) = \{1,0.5,0.5\}$

Note that every mset $M \in \mathfrak{M}(\mathcal{S})$ is a fuzzy mset since if $S = \{x_1, x_2, \dots, x_n\}$ and
$M = \{m_1/x_1, m_2/x_2, \dots, m_n/x_n\}$ then M can be rewritten :

$$M = \left\{ \begin{array}{c} \overbrace{(x_1,1),(x_1,1),\dots,(x_1,1)}^{m_1}, \overbrace{(x_2,1),(x_2,1),\dots,(x_2)}^{m_2},\dots, \\ \overbrace{(x_n,1),(x_n,1),\dots,(x_n 1)}^{m_n} \end{array} \right\}$$

$$= \left\{ \overbrace{\{1,1,\dots,1\}}^{m_1} \Big/ x_1, \overbrace{\{1,1,\dots,1\}}^{m_2} \Big/ x_2, \dots, \overbrace{\{1,1,\dots,1\}}^{m_n} \Big/ x_n \right\}$$

Here, $\Psi_M(x_i) = \overbrace{\{1,1,\dots,1\}}^{m_i}$ for all $i \in [1,n]$.

The class of all finite fuzzy msets drawn from the set $\mathcal{S}$ shall be denoted by $\mathcal{FM}(\mathcal{S})$.
A fuzzy mset $M \in \mathcal{FM}(\mathcal{S})$ is empty denoted $\emptyset$ if and only if $\Psi_\emptyset(x) = \{0,0,\dots,0\}$ for all $x \in \mathcal{S}$.
For any $M \in \mathcal{FM}(\mathcal{S})$, $x \in M \leftrightarrow \Psi_M(x) \neq \Psi_\emptyset(x)$.
The root (Support) set of a fuzzy mset $M \in \mathcal{FM}(\mathcal{S})$ denoted M^* is defined:

$$M^* = \{x \in \mathcal{S} | \Psi_M(x) \neq \Psi_\emptyset(x)\}$$

Notes that for any $M, N \in \mathcal{FM}(\mathcal{S})$,
$\Psi_{M+N}(x) = \Psi_M(x) + \Psi_N(x)$,
$\mathrm{M} = \mathrm{N} \leftrightarrow \Psi_M(x) = \Psi_N(x)$ and
$M \subseteq N \leftrightarrow \Psi_M(x) \subseteq \Psi_N(x)$ for all $x \in \mathcal{S}$ where $+$ and $\subseteq$ are msets addition and inclusion relations respectively.

Definition 2.4 (membership sequence). The membership sequence for an element $x \in S$ in the fuzzy mset M is defined to be the decreasing ordered elements of $\Psi_M(x)$. It is denoted by $\Psi_{\langle M \rangle}(x)$ where $\Psi_{\langle M \rangle}(x)$ is defined:

$$\Psi_{\langle M \rangle}(x) = \langle \Psi_M(x) \rangle = \left(\mu_M^1(x), \mu_M^2(x), \dots, \mu_M^p(x)\right) \text{ such that } \mu_M^1(x) \geq \mu_M^2(x) \geq \dots \geq \mu_M^p(x).$$

For example if $A = \{\{0.2, 0.3\}/a, \{0.3, 0, 2, 1, 0.5, 0.5\}/b\}$, then
$\Psi_{\langle A \rangle}(a) = (0.3, 0.2)$ and $\Psi_{\langle A \rangle}(b) = (1, 0.5, 0.5, 0.3, 0.2)$.
Notes that $\Psi_{\langle M \rangle}(x) = \left(\mu_M^1(x), \mu_M^2(x), \dots, \mu_M^p(x)\right)$ implies that
$\Psi_M(x) = \left\{\mu_M^1(x), \mu_M^2(x), \dots, \mu_M^p(x)\right\}$.
When an operation between two fuzzy msets M and N is defined, the lengths of the membership sequence $\Psi_{\langle M \rangle}(x) = \left(\mu_M^1(x), \mu_M^2(x), \dots, \mu_M^p(x)\right)$, and
$\Psi_{\langle N \rangle}(x) = \left(\mu_N^1(x), \mu_N^2(x), \dots, \mu_N^{p'}(x)\right)$ are set to be equal.
Note that for an empty fuzzy mset $\emptyset \in \mathcal{FM}(\mathcal{S})$,
$\Psi_{\langle \emptyset \rangle}(x) = (0,0,\dots,0)$. In particular, $\mu_\emptyset^i(x) = 0$ for all i.

Definition 2.5. Let $M \in \mathcal{FM}(\mathcal{S})$. The set
$\langle M \rangle = \left\{\Psi_{\langle M \rangle}(x)/x \,\middle|\, x \in \mathcal{S}\right\}$
Is called the membership sequenced fuzzy mset for M

Definition 2.6 [19] **(Length of a membership sequence).** Let $M, N \in \mathcal{FM}(\mathcal{S})$. The length of the membership sequence $\Psi_{\langle M \rangle}(x)$ denoted $L(x; M)$ is defined:

$$L(x; M) = max\left\{j : \mu_M^j(x) > 0\right\} \text{ and } L(x; M, N) = max\{L(x; M), L(x; N)\}$$

However,

$$L(x;\emptyset)=0 \text{ and } L(x;\emptyset,N)=L(x;N,\emptyset)=L(x;N)$$

When no ambiguity arises, $L(x)=L(x;M,N)$.
For example, let $M=\{\{0.2,0.3\}/x,\{1,0.5,0.5\}/y\}$ $N=\{\{0.6\}/x,\{0.8,0.6\}/y,\{0.1,0.7\}/w\}$
From $\Psi_{\langle M\rangle}(x)$ $\Psi_{\langle N\rangle}(x),\Psi_{\langle M\rangle}(y),\Psi_{\langle N\rangle}(y),\Psi_{\langle M\rangle}(w)$
and $\Psi_{\langle N\rangle}(w)$, $L(x)=2$, $L(y)=3, L(z)=0, L(w)=2$ and
$M=\{\{0.3,0.2\}/x,\{1,0.5,0.5\}/y,\{0,0\}/w\}$, $N=\{\{0.6,0\}/x,\{0.8,0.6,0\}/y,\{0.7,0.1\}/w\}$
Definition 2.7. For any msets $M,N\in\mathcal{FM}(\mathcal{S})$ the following hold:

(i). $\Psi_{\langle M\rangle}(x)\preccurlyeq\Psi_{\langle N\rangle}(x)$ if and only if $L(x;M)\le L(x;N)$ and $\mu_M^i(x)\le\mu_N^i(x)$, $i=1,2,\dots,L(x;N)$, for all $x\in\mathcal{S}$

Note that $\Psi_{\langle M\rangle}(x)\prec\Psi_{\langle N\rangle}(x)$ if and only if $\mu_M^i(x)<\mu_N^i(x)$, for some $i\in[1,L(x;N)]$, for all $x\in\mathcal{S}$
(ii). $\Psi_{\langle M\rangle}(x)\vee\Psi_{\langle N\rangle}(x)=\left(\mu_M^i(x)\vee\mu_N^i(x)\middle|\, i=1,2,\dots,L(x)\right)$ for all $x\in\mathcal{S}$
Where $\mu_M^i(x)\vee\mu_N^i(x)=max\{\mu_M^i(x),\mu_N^i(x)\}$
(iii). $\Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle N\rangle}(x)=\left(\mu_M^i(x)\wedge\mu_N^i(x)\middle|\, i=1,2,\dots,L(x)\right)$ for all $x\in\mathcal{S}$
Where $\mu_M^i(x)\wedge\mu_N^i(x)=min\{\mu_M^i(x),\mu_N^i(x)\}$

(iv). $\Psi_{\langle M\rangle}(x)-\Psi_{\langle N\rangle}(x)=\left\{\mu_M^i(x)-\mu_N^i(x)\vee 0\middle|\, i=1,2,\dots,L(x)\right\}$ for all $x\in\mathcal{S}$
(v). $\Psi_{\langle M\rangle}(x)\uplus\Psi_{\langle N\rangle}(x)=\Psi_{\langle M+N\rangle}(x)$ for all $x\in\mathcal{S}$
(vi). $\Psi_{\langle M\rangle}(x)+\Psi_{\langle N\rangle}(x)=\left(\mu_M^i(x)+\mu_N^i(x)-\mu_M^i(x).\mu_N^i(x)\middle|\, i=1,2,\dots,L(x)\right)$ for all $x\in\mathcal{S}$
(vii). $\Psi_{\langle M\rangle}(x).\Psi_{\langle N\rangle}(x)=\left(\mu_M^i(x).\mu_N^i(x)\middle|\, i=1,2,\dots,L(x)\right)$ for all $x\in\mathcal{S}$

3.0 Operations and Algebraic properties on fuzzy multisets with membership sequence

Definition 3.1. For any $M,N\in\mathcal{FM}(\mathcal{S})$, the basic relations and operations on membership sequenced fuzzy msets $\langle M\rangle$ and $\langle N\rangle$ from the literature are defined:

(i). (inclusion) : $\langle M\rangle\subseteq\langle N\rangle\leftrightarrow\Psi_{\langle M\rangle}(x)\preccurlyeq\Psi_{\langle N\rangle}(x)$, for all $x\in\mathcal{S}$.
(ii). (equality): $\langle M\rangle=\langle N\rangle\leftrightarrow\Psi_{\langle M\rangle}(x)=\Psi_{\langle N\rangle}(x)$ for all $x\in\mathcal{S}$.
(iii). (Union): $\Psi_{\langle M\rangle\cup\langle N\rangle}(x)=\Psi_{\langle M\rangle}(x)\vee\Psi_{\langle N\rangle}(x)$ for all $x\in\mathcal{S}$.
(v). (additive Union): $\Psi_{\langle M\rangle\uplus\langle N\rangle}(x)=\Psi_{\langle M\rangle}(x)\uplus\Psi_{\langle N\rangle}(x)$ for all $x\in\mathcal{S}$.
(vi). (addition): $\Psi_{\langle M\rangle\oplus\langle N\rangle}(x)=\Psi_{\langle M\rangle}(x)+\Psi_{\langle N\rangle}(x)$ for all $x\in\mathcal{S}$
(vii). (Multiplication): $\Psi_{\langle M\rangle\otimes\langle N\rangle}(x)=\Psi_{\langle M\rangle}(x).\Psi_{\langle N\rangle}(x)$ for all $x\in\mathcal{S}$
(viii). (intersection): $\Psi_{\langle M\rangle\cap\langle N\rangle}(x)=\Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle N\rangle}(x)$ for all $x\in\mathcal{S}$.
(ix). (difference): $\Psi_{\langle M\rangle-\langle N\rangle}(x)=\Psi_{\langle M\rangle}(x)-\Psi_{\langle N\rangle}(x)$ for all $x\in\mathcal{S}$.
(x). (complement): $\Psi_{\langle M\rangle^c}(x)=\Psi_{\langle\mathfrak{U}\rangle-\langle M\rangle}(x)$ for all $x\in\mathcal{S}$
Where $\mathfrak{U}$ is a mset such that $|\Psi_M(x)|=C_{\mathfrak{U}}(x)$

Note that $\mu_{\mathfrak{U}}^i(x)=1$ and $\mu_{\mathfrak{U}}^i(x)-\mu_M^i(x)\ge 0$
In particular, $\mu_{\mathfrak{U}}^i(x)-\mu_M^i(x)\vee 0=\mu_{\mathfrak{U}}^i(x)-\mu_M^i(x)$.

i.e $\Psi_{\langle M\rangle^c}(x) = \left(1-\mu_M^i(x)\middle| i = 1,2,\dots,L(x;M)\right)$.
Let $1-\mu_M^i(x) = \mu_{M'}^i(x)$ so that $\Psi_{\langle M\rangle^c}(x) = \left(\mu_{M'}^i(x)\middle| i = 1,2,\dots,L(x;M)\right)$

Proposition 3.1. Let us suppose that $M, N \in \mathcal{FM}(\mathcal{S})$, then the following properties hold:

(i) $\langle M\rangle = \langle N\rangle \leftrightarrow M = N$
(ii) $\Psi_{\langle M\rangle}(x) \preccurlyeq \Psi_{\langle N\rangle}(x) \wedge \Psi_{\langle N\rangle}(x) \preccurlyeq \Psi_{\langle M\rangle}(x) \rightarrow \Psi_{\langle M\rangle}(x) = \Psi_{\langle N\rangle}(x)$ for all $x \in \mathcal{S}$
(iii) $\langle M\rangle \subseteq \langle N\rangle \wedge \langle N\rangle \subseteq \langle M\rangle \rightarrow \langle M\rangle = \langle N\rangle$

Proof:
(i). if $\langle M\rangle = \langle N\rangle$, we have $\Psi_{\langle M\rangle}(x) = \Psi_{\langle N\rangle}(x)$ for all $x \in \mathcal{S}$(by definition)
Thus, $\langle\Psi_M(x)\rangle = \langle\Psi_N(x)\rangle$. In particular, $\Psi_M(x) = \Psi_N(x)$ for all $x \in \mathcal{S}$ and $M = N$
Conversely, let $M = N$. We have $\Psi_M(x) = \Psi_N(x)$ for all $x \in \mathcal{S}$ (by definition).
In particular, $\langle\Psi_M(x)\rangle = \langle\Psi_N(x)\rangle$ and $\Psi_{\langle M\rangle}(x) = \Psi_{\langle N\rangle}(x)$.
Thus, $\langle M\rangle = \langle N\rangle$ (by definition)

(ii). Assuming $\Psi_{\langle M\rangle}(x) \preccurlyeq \Psi_{\langle N\rangle}(x)$. We have
$L(x;M) \le L(x;N)$ (1) and $\mu_M^i(x) \le \mu_N^i(x),\ i = 1,2,\dots,L(x;N)$, (2)
Now $\Psi_{\langle N\rangle}(x) \preccurlyeq \Psi_{\langle M\rangle}(x)$ implies
$L(x;N) \le L(x;M)$ (3) and $\mu_N^i(x) \le \mu_M^i(x),\ i = 1,2,\dots,L(x;M)$ (4)
From (1) and (2), we have $L(x;M) = L(x;N)$
From (2) and (4), we have $\mu_M^i(x) = \mu_N^i(x),\ i = 1,2,\dots,L(x;M) = L(x;N)$
In particular, $\Psi_{\langle M\rangle}(x) = \Psi_{\langle N\rangle}(x)$
(iii). Now $\langle M\rangle \subseteq \langle N\rangle \wedge \langle N\rangle \subseteq \langle M\rangle \rightarrow \Psi_{\langle M\rangle}(x) \preccurlyeq \Psi_{\langle N\rangle}(x) \wedge \Psi_{\langle N\rangle}(x) \preccurlyeq \Psi_{\langle M\rangle}(x)$
Hence, $\Psi_{\langle M\rangle}(x) = \Psi_{\langle N\rangle}(x)$ for all $x \in \mathcal{S}$.
In particular, $\langle M\rangle = \langle N\rangle$.

For any fuzzy mset $M \in \mathcal{FM}(\mathcal{S})$, we denote the root (support) set of it membership sequence by $\langle M\rangle^*$ define:
$\langle M\rangle^* = \left\{x \in \mathcal{S}\middle|\Psi_{\langle M\rangle}(x) \succ \Psi_{\langle\emptyset\rangle}(x)\right\}$

Definition 3.2. Let $M, N \in \mathcal{FM}(\mathcal{S})$. The membership sequenced fuzzy mset $\langle M\rangle$ is said to be a whole fuzzy submset of the membership sequenced fuzzy mset $\langle N\rangle$ if $\langle M\rangle \subseteq \langle N\rangle$ and $\mu_M^i(x) = \mu_N^i(x)$ for all $i = 1,2,\dots,L(x{:}M)$
and $\langle M\rangle$ is said to be a full fuzzy submset of $\langle N\rangle$ if $\langle M\rangle \subseteq \langle N\rangle$ and $L(x{:}M) = L(x{:}N) = L(x)$.
The powerset of a sequenced fuzzy mset $\langle N\rangle$ denoted by $\wp(\langle N\rangle)$ is defined: $\wp(\langle N\rangle) = \{\langle M\rangle|\langle M\rangle \subseteq \langle N\rangle\}$
The power-whole set of $\langle N\rangle$ denoted by $\wp w(\langle N\rangle)$ is defined:
$\wp w(\langle N\rangle) = \{\langle M\rangle|\langle M\rangle$ is a whole fuzzy submset of $\langle N\rangle\}$ and
The powerfull set $\langle N\rangle$ denoted by $\wp f(\langle N\rangle)$ is defined:
$\wp f(\langle N\rangle) = \{\langle M\rangle|\langle M\rangle$ is a full fuzzy submset of $\langle N\rangle\}$

Clearly, $\langle\emptyset\rangle, \langle N\rangle \in \wp f(\langle N\rangle)$ and $N \in \wp w(\langle N\rangle)$ but $\emptyset \notin \wp w(\langle N\rangle)$.

Proposition 3.2. If $N \in \mathcal{FM}(\mathcal{S})$, then any
$\langle M\rangle, \langle P\rangle \in \wp w(\langle N\rangle)$, we have:
(i) $\langle M\rangle \cup \langle P\rangle \in \wp w(\langle N\rangle)$ (ii). $\langle M\rangle \cap \langle P\rangle \in \wp w(\langle N\rangle)$

Proof:
(i).Now $\langle M\rangle, \langle P\rangle \in \wp w(\langle N\rangle)$ implies that
$\langle M\rangle \subseteq \langle N\rangle$ and $\mu_M^i(x) = \mu_N^i(x)$ for all $i \in [1, L(x; M)]$,
$\langle P\rangle \subseteq \langle N\rangle$ and $\mu_P^i(x) = \mu_N^i(x)$ for all $i \in [1, L(x; P)]$.
Now $\mu_M^i(x) \vee \mu_P^i(x) = \mu_N^i(x)$ for all $i \in [1, L(x)]$ and $L(x) \leq L(x; N)$.
Thus, $\langle M\rangle \cup \langle P\rangle \in \wp w(\langle N\rangle)$
(ii).We show that $\langle M\rangle \cap \langle P\rangle \in \wp w(\langle N\rangle)$
Now, $\mu_M^i(x) \wedge \mu_P^i(x) = \mu_N^i(x)$ for all $i \in [1, L(x)]$
and $L(x) \leq L(x; N)$. Hence,
$\langle M\rangle \cap \langle P\rangle \in \wp w(\langle N\rangle)$

Proposition 3.3 . If $N \in \mathcal{FM}(\mathcal{S})$, then any
$\langle M\rangle, \langle P\rangle \in \wp f(\langle N\rangle)$, we have:
(i) $\langle M\rangle \cup \langle P\rangle \in \wp f(\langle N\rangle)$ (ii). $\langle M\rangle \cap \langle P\rangle \in \wp f(\langle N\rangle)$

Proof:
(i). For $\langle M\rangle, \langle P\rangle \in \wp f(\langle N\rangle)$, we show that
$\langle M\rangle \cup \langle P\rangle \in \wp f(\langle N\rangle)$.
Now since $\langle M\rangle, \langle P\rangle \in \wp f(\langle N\rangle)$, we have
$\langle M\rangle \subseteq \langle N\rangle, L(x\colon M) = L(x\colon N)$ (1) and
$\langle P\rangle \subseteq \langle N\rangle, L(x\colon P) = L(x\colon N)$ (2)
Now $\mu_M^i(x) \vee \mu_P^i(x) \leq \mu_N^i(x)$ for all $i \in [1, L(x)]$
(since $\mu_M^i(x), \mu_P^i(x) \leq \mu_N^i(x)$)
From (1) and (2), $L(x) = L(x\colon N)$
Thus, $\langle M\rangle \cup \langle P\rangle \in \wp f(\langle N\rangle)$.
(ii). Now, $\mu_M^i(x) \wedge \mu_P^i(x) \leq \mu_N^i(x)$ for all $i \in [1, L(x)]$
(since $\mu_M^i(x), \mu_P^i(x) \leq \mu_N^i(x)$)
From (1) and (2), $L(x) = L(x\colon N)$
Thus, $\langle M\rangle \cap \langle P\rangle \in \wp f(\langle N\rangle)$

Proposition 3.4. Let $M, N \in \mathcal{FM}(\mathcal{S})$. The following are decreasing membership sequence for any $x \in \mathcal{S}$.

(i). $\Psi_{\langle M\rangle \cup \langle N\rangle}(x)$ (ii). $\Psi_{\langle M\rangle \cap \langle N\rangle}(x)$ (iii). $\Psi_{\langle M\rangle \uplus \langle N\rangle}(x)$

Proof:
(i). Let $\Psi_{\langle M\rangle}(x) = \left(\mu_M^1(x), \mu_M^2(x), \ldots, \mu_M^p(x)\right)$ and $\Psi_{\langle N\rangle}(x) = \left(\mu_N^1(x), \mu_N^2(x), \ldots, \mu_N^{p'}(x)\right)$ such that $\mu_M^1(x) \geq \mu_M^2(x) \geq \cdots \geq \mu_M^p(x)$ and $\mu_N^1(x) \geq \mu_N^2(x) \geq \cdots \geq \mu_N^{p'}(x)$ respectively.
We show that
$\mu_{\langle M\rangle \cup \langle N\rangle}^1(x) \geq \mu_{\langle M\rangle \cup \langle N\rangle}^2(x) \geq \cdots \geq \mu_{\langle M\rangle \cup \langle N\rangle}^{L(x)}(x)$
Where $\mu_{\langle M\rangle \cup \langle N\rangle}^i(x) = \mu_M^i(x) \vee \mu_N^i(x), i = 1,2, \ldots, L(x)$.
Now we prove by induction starting with $i = 1$ and $i = 2$
To show that $\mu_M^1(x) \vee \mu_N^1(x) \geq \mu_M^2(x) \vee \mu_N^2(x)$.
But $\mu_M^1(x) \vee \mu_N^1(x) \geq \mu_M^2(x)$ and
$\mu_M^1(x) \vee \mu_N^1(x) \geq \mu_N^2(x)$ (by definition above).

Thus, $\mu_M^1(x) \vee \mu_N^1(x) \geq \mu_M^2(x) \vee \mu_N^2(x)$

In particular, $\mu^{1}_{\langle M\rangle\cup\langle N\rangle}(x) \geq \mu^{2}_{\langle M\rangle\cup\langle N\rangle}(x)$.

Now assuming it is true for $i = j$ and $i = j + 1$
In particular, $\mu^{j}_{\langle M\rangle\cup\langle N\rangle}(x) \geq \mu^{j+1}_{\langle M\rangle\cup\langle N\rangle}(x)$
i.e $\mu^{j}_{M}(x) \vee \mu^{j}_{N}(x) \geq \mu^{j+1}_{M}(x) \vee \mu^{j+1}_{N}(x)$.

We show that it is true for $i = k + 1$ and $i = k + 2$
i.e $\mu^{k+1}_{\langle M\rangle\cup\langle N\rangle}(x) \geq \mu^{k+2}_{\langle M\rangle\cup\langle N\rangle}(x)$ (1)

Now let $k + 1 = j$. But $k + 2 = (k + 1) + 1$
Thus, $\mu^{k+1}_{\langle M\rangle\cup\langle N\rangle}(x) = \mu^{j}_{\langle M\rangle\cup\langle N\rangle}(x)$ and
$\mu^{k+2}_{\langle M\rangle\cup\langle N\rangle}(x) = \mu^{(k+1)+1}_{\langle M\rangle\cup\langle N\rangle}(x) = \mu^{j+1}_{\langle M\rangle\cup\langle N\rangle}(x)$

Hence, $\mu^{k+1}_{\langle M\rangle\cup\langle N\rangle}(x) \geq \mu^{k+2}_{\langle M\rangle\cup\langle N\rangle}(x)$ (by hypothesis).
In particular,

$$\mu^{1}_{\langle M\rangle\cup\langle N\rangle}(x) \geq \mu^{2}_{\langle M\rangle\cup\langle N\rangle}(x) \geq \cdots \geq \mu^{L(x)}_{\langle M\rangle\cup\langle N\rangle}(x)$$

(ii). We show that

$$\mu^{1}_{\langle M\rangle\cap\langle N\rangle}(x) \geq \mu^{2}_{\langle M\rangle\cap\langle N\rangle}(x) \geq \cdots \geq \mu^{L(x)}_{\langle M\rangle\cap\langle N\rangle}(x)$$

Now, $\mu^{2}_{M}(x) \wedge \mu^{2}_{N}(x) \leq \mu^{2}_{M}(x) \leq \mu^{1}_{M}(x)$ and
$\mu^{2}_{M}(x) \wedge \mu^{2}_{N}(x) \leq \mu^{2}_{N}(x) \leq \mu^{1}_{N}(x)$ (by definition)

Thus, $\mu^{2}_{M}(x) \wedge \mu^{2}_{N}(x) \leq \mu^{1}_{M}(x) \wedge \mu^{1}_{N}(x)$
In particular, $\mu^{1}_{\langle M\rangle\cap\langle N\rangle}(x) \geq \mu^{2}_{\langle M\rangle\cap\langle N\rangle}(x)$.
Now assuming it is true for $i = j$ and $i = j + 1$
In particular, $\mu^{j}_{\langle M\rangle\cap\langle N\rangle}(x) \geq \mu^{j+1}_{\langle M\rangle\cap\langle N\rangle}(x)$
i.e $\mu^{j}_{M}(x) \wedge \mu^{j}_{N}(x) \geq \mu^{j+1}_{M}(x) \wedge \mu^{j+1}_{N}(x)$.

We show that it is true for $i = k + 1$ and $i = k + 2$
i.e $\mu^{k+1}_{\langle M\rangle\cap\langle N\rangle}(x) \geq \mu^{k+2}_{\langle M\rangle\cap\langle N\rangle}(x)$ (1)
Now let $k + 1 = j$. But $k + 2 = (k + 1) + 1$

Thus, $\mu^{k+1}_{\langle M\rangle\cap\langle N\rangle}(x) = \mu^{j}_{\langle M\rangle\cap\langle N\rangle}(x)$ and
$\mu^{k+2}_{\langle M\rangle\cap\langle N\rangle}(x) = \mu^{(k+1)+1}_{\langle M\rangle\cap\langle N\rangle}(x) = \mu^{j+1}_{\langle M\rangle\cap\langle N\rangle}(x)$

Hence, $\mu^{k+1}_{\langle M\rangle\cap\langle N\rangle}(x) \geq \mu^{k+2}_{\langle M\rangle\cap\langle N\rangle}(x)$ (by hypothesis).
In particular,

$$\mu^{1}_{\langle M\rangle\cap\langle N\rangle}(x) \geq \mu^{2}_{\langle M\rangle\cap\langle N\rangle}(x) \geq \cdots \geq \mu^{L(x)}_{\langle M\rangle\cap\langle N\rangle}(x).$$

(iii) the result is clear by definition.

Notes that $\langle \Psi_{\langle M\rangle}(x)\rangle = \Psi_{\langle M\rangle}(x)$,

$\langle\Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle N\rangle}(x)\rangle = \Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle N\rangle}(x)$ and
$\langle\Psi_{\langle M\rangle}(x)\vee\Psi_{\langle N\rangle}(x)\rangle = \Psi_{\langle M\rangle}(x)\vee\Psi_{\langle N\rangle}(x)$.

Theorem 3.5 [18]. Let $M, N, P \in \mathcal{FM}(\mathcal{S})$. The following equalities are valid:

(i).$\langle M\rangle\cup\langle N\rangle = \langle N\rangle\cup\langle M\rangle$
(ii). $\langle M\rangle\cap\langle N\rangle = \langle N\rangle\cap\langle M\rangle$
(iii). $\langle M\rangle\cup(\langle N\rangle\cup\langle P\rangle) = (\langle M\rangle\cup\langle N\rangle)\cup\langle P\rangle$
(iv). $\langle M\rangle\cap(\langle N\rangle\cap\langle P\rangle) = (\langle M\rangle\cap\langle N\rangle)\cap\langle P\rangle$
(v). $\langle M\rangle\cup(\langle N\rangle\cap\langle P\rangle) = (\langle M\rangle\cup\langle N\rangle)\cap(\langle M\rangle\cup\langle P\rangle)$
(vi). $\langle M\rangle\cap(\langle N\rangle\cup\langle P\rangle) = (\langle M\rangle\cap\langle N\rangle)\cup(\langle M\rangle\cap\langle P\rangle)$

Proposition 3.6. For any $M, N, P \in \mathcal{FM}(\mathcal{S})$, the following results holds:

(i). $\langle M\rangle\cup\langle M\rangle = \langle M\rangle$
(ii) $\langle M\rangle\cap\langle M\rangle = \langle M\rangle$
(iii). $\langle M\rangle\cup\langle\emptyset\rangle = \langle M\rangle$
(iv). $\langle M\rangle\cap\langle\emptyset\rangle = \langle\emptyset\rangle$
(v). $\langle M\rangle\cup(\langle M\rangle\cap\langle N\rangle) = \langle M\rangle$
(vi). $\langle M\rangle\cap(\langle M\rangle\cup\langle N\rangle) = \langle M\rangle$
(vii). $\langle M\rangle \uplus (\langle N\rangle \uplus \langle P\rangle) = (\langle M\rangle \uplus \langle N\rangle) \uplus \langle P\rangle$
(viii). $\langle M\rangle \uplus (\langle N\rangle\cup\langle P\rangle) = (\langle M\rangle \uplus \langle N\rangle)\cup(\langle M\rangle \uplus \langle P\rangle)$
(ix). $\langle M\rangle \uplus (\langle N\rangle\cap\langle P\rangle) = (\langle M\rangle \uplus \langle N\rangle)\cap(\langle M\rangle \uplus \langle P\rangle)$
(x). $\langle M\rangle \uplus \langle N\rangle = \langle M\cup N\rangle \uplus \langle M\cap N\rangle$.

Proof:

(i). $\Psi_{\langle M\rangle\cup\langle M\rangle}(x) = \Psi_{\langle M\rangle}(x)\vee\Psi_{\langle M\rangle}(x) = \left(\mu_M^i(x) \vee \mu_M^i(x)\middle| i = 1,2, \dots, L(x{:}M)\right)$
But $\mu_M^i(x) \vee \mu_M^i(x) = \mu_M^i(x)$ for all $x \in \mathcal{S}$ and $i = 1,2, \dots, L(x{:}M)$
Thus, $\Psi_{\langle M\rangle}(x)\vee\Psi_{\langle M\rangle}(x) = \left(\mu_M^i(x)\middle| i = 1,2, \dots, L(x{:}M)\right) = \Psi_{\langle M\rangle}(x)$
In particular, $\Psi_{\langle M\rangle\cup\langle M\rangle}(x) = \Psi_{\langle M\rangle}(x)$
Thus, $\langle M\rangle\cup\langle M\rangle = \langle M\rangle$

(ii). $\Psi_{\langle M\rangle\cap\langle M\rangle}(x) = \Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle M\rangle}(x) = \left(\mu_M^i(x) \wedge \mu_M^i(x)\middle| i = 1,2, \dots, L(x{:}M)\right)$.
But $\mu_M^i(x) \wedge \mu_M^i(x) = \mu_M^i(x)$ for all $x \in \mathcal{S}$ and $i = 1,2, \dots, L(x{:}M)$
Thus, $\Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle M\rangle}(x) = \left(\mu_M^i(x)\middle| i = 1,2, \dots, L(x{:}M)\right) = \Psi_{\langle M\rangle}(x)$
In particular, $\Psi_{\langle M\rangle\cap\langle M\rangle}(x) = \Psi_{\langle M\rangle}(x)$
Thus, $\langle M\rangle\cap\langle M\rangle = \langle M\rangle$

(iii). $\Psi_{\langle M\rangle\cup\langle\emptyset\rangle}(x) = \Psi_{\langle M\rangle}(x)\vee\Psi_{\langle\emptyset\rangle}(x) = \left(\mu_M^i(x) \vee 0\middle| i = 1,2, \dots, L(x{:}M)\right)$
But $\mu_M^i(x) \vee 0 = \mu_M^i(x)$
Thus, $\Psi_{\langle M\rangle}(x)\vee\Psi_{\langle\emptyset\rangle}(x) = \left(\mu_M^i(x)\middle| i = 1,2, \dots, L(x{:}M)\right) = \Psi_{\langle M\rangle}(x)$
In particular, $\Psi_{\langle M\rangle\cup\langle\emptyset\rangle}(x) = \Psi_{\langle M\rangle}(x)$
Thus, $\langle M\rangle\cup\langle\emptyset\rangle = \langle M\rangle$

(iv). $\Psi_{\langle M\rangle\cap\langle\emptyset\rangle}(x) = \Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle\emptyset\rangle}(x)$
$= \left(\mu_M^i(x) \wedge 0\middle| i = 1,2, \dots, L(x{:}M)\right)$
$= \left(0\middle| i = 1,2, \dots, L(x{:}M)\right)$

$= \Psi_{\langle\emptyset\rangle}(x)$

Thus, $\Psi_{\langle M\rangle\cap\langle\emptyset\rangle}(x) = \Psi_{\langle\emptyset\rangle}(x)$

In particular, $\langle M\rangle\cap\langle\emptyset\rangle = \langle\emptyset\rangle$

(v). $\Psi_{\langle M\rangle\cup(\langle M\rangle\cap\langle N\rangle)}(x) = \Psi_{\langle M\rangle}(x)\vee\Psi_{\langle M\rangle\cap\langle N\rangle}(x)$

$= \Psi_{\langle M\rangle}(x)\vee\left(\Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle N\rangle}(x)\right)\left(\mu_M^i(x)\vee\left(\mu_M^i(x)\wedge\mu_N^i(x)\right)\middle|i = 1,2,\ldots,L(x)\right)$

$= \left(\mu_M^i(x)\middle|i = 1,2,\ldots,L(x{:}\,M)\right)$

$= \Psi_{\langle M\rangle}(x)$

Thus, $\Psi_{\langle M\rangle\cup(\langle M\rangle\cap\langle N\rangle)}(x) = \Psi_{\langle M\rangle}(x)$

In particular, $\langle M\rangle\cup(\langle M\rangle\cap\langle N\rangle) = \langle M\rangle$

(vi). $\Psi_{\langle M\rangle\cap(\langle M\rangle\cup\langle N\rangle)}(x) = \Psi_{\langle M\rangle}(x)\wedge\Psi_{\langle M\rangle\cup\langle N\rangle}(x)$

$= \Psi_{\langle M\rangle}(x)\wedge\left(\Psi_{\langle M\rangle}(x)\vee\Psi_{\langle N\rangle}(x)\right)$

$= \left(\mu_M^i(x)\wedge\left(\mu_M^i(x)\vee\mu_N^i(x)\right)\middle|i = 1,2,\ldots,L(x)\right)$

$= \left\{\mu_M^i(x)\middle|i = 1,2,\ldots,L(x{:}\,M)\right\} = \Psi_{\langle M\rangle}(x)$

Thus, $\Psi_{\langle M\rangle\cap(\langle M\rangle\cup\langle N\rangle)}(x) = \Psi_{\langle M\rangle}(x)$

In particular, $\langle M\rangle\cap(\langle M\rangle\cup\langle N\rangle) = \langle M\rangle$

(vii) $\Psi_{\langle M\rangle\uplus(\langle N\rangle\uplus\langle P\rangle)}(x) = \Psi_{\langle M\rangle}(x)\uplus\Psi_{\langle N\rangle\uplus\langle P\rangle}(x)$

$= \Psi_{\langle M\rangle}(x)\uplus\left(\Psi_{\langle N\rangle}(x)\uplus\Psi_{\langle P\rangle}(x)\right)$

$= \Psi_{\langle M+(N+P)\rangle}(x) = \langle\Psi_{M+(N+P)}(x)\rangle$

$= \langle\Psi_M(x) + \Psi_{N+P}(x)\rangle$

$= \langle\Psi_M(x) + \left(\Psi_N(x) + \Psi_P(x)\right)\rangle$

$= \langle\left(\Psi_M(x) + \Psi_N(x)\right) + \Psi_P(x)\rangle$

$= \langle\Psi_{M+N}(x) + \Psi_P(x)\rangle = \langle\Psi_{(M+N)+P}(x)\rangle$

$= \Psi_{\langle(M+N)+P\rangle}(x) = \left(\Psi_{\langle M\rangle}(x)\uplus\Psi_{\langle N\rangle}(x)\right)\uplus\Psi_{\langle P\rangle}(x)$

$= \Psi_{\langle M\rangle\uplus\langle N\rangle}(x)\uplus\Psi_{\langle P\rangle}(x) = \Psi_{(\langle M\rangle\uplus\langle N\rangle)\uplus\langle P\rangle}(x)$

Thus, $\Psi_{\langle M\rangle\uplus(\langle N\rangle\uplus\langle P\rangle)}(x) = \Psi_{(\langle M\rangle\uplus\langle N\rangle)\uplus\langle P\rangle}(x)$

In particular, $\langle M\rangle\uplus(\langle N\rangle\uplus\langle P\rangle) = (\langle M\rangle\uplus\langle N\rangle)\uplus\langle P\rangle$

(viii).$\Psi_{\langle M\rangle\uplus(\langle N\rangle\cup\langle P\rangle)}(x) = \Psi_{\langle M\rangle}(x)\uplus\Psi_{\langle N\rangle\cup\langle P\rangle}(x)$

$= \Psi_{\langle M\rangle}(x)\uplus\left(\Psi_{\langle N\rangle}(x)\vee\Psi_{\langle P\rangle}(x)\right)$

$= \Psi_{\langle M+Q\rangle}(x)$ where $\langle Q\rangle = \langle N\rangle\cup\langle P\rangle$ and

$= \langle\Psi_{M+Q}(x)\rangle = \langle\Psi_M(x) + \Psi_Q(x)\rangle$ where

$\Psi_Q(x) = \left\{\mu_N^i(x)\vee\mu_P^i(x)\middle|i = 1,2,\ldots,L(x)\right\}$

Thus, $\langle\Psi_{M+Q}(x)\rangle$

$= \langle\Psi_M(x) + \left\{\mu_N^i(x)\vee\mu_P^i(x)\middle|i = 1,2,\ldots,L(x)\right\}\rangle$

But $\langle\Psi_M(x) + \left\{\mu_N^i(x)\vee\mu_P^i(x)\middle|i = 1,2,\ldots,L(x)\right\}\rangle$

$$= \left\langle \begin{matrix}\langle\Psi_M(x) + \left\{\mu_N^i(x)\middle|i = 1,2,\ldots,L(x{:}\,N)\right\}\rangle \\ \vee\langle\Psi_M(x) + \left\{\mu_P^i(x)\middle|i = 1,2,\ldots,L(x{:}\,P)\right\}\rangle\end{matrix}\right\rangle$$

$= \langle\langle\Psi_M(x) + \Psi_N(x)\rangle\vee\langle\Psi_M(x) + \Psi_P(x)\rangle\rangle$

$= \Psi_{\langle M+N\rangle}(x)\vee\Psi_{\langle M+P\rangle}(x)$

$= \Psi_{\langle M\rangle\uplus\langle N\rangle}(x)\vee\Psi_{\langle M\rangle\uplus\langle P\rangle}(x)$

$= \Psi_{(\langle M\rangle\uplus\langle N\rangle)\cup(\langle M\rangle\uplus\langle P\rangle)}(x)$

In particular,
$\langle M\rangle \uplus (\langle N\rangle \cup \langle P\rangle) = (\langle M\rangle \uplus \langle N\rangle)\cup(\langle M\rangle \uplus \langle P\rangle)$

(ix). $\Psi_{\langle M\rangle \uplus (\langle N\rangle \cap \langle P\rangle)}(x) = \Psi_{\langle M\rangle}(x) \uplus \Psi_{\langle N\rangle \cap \langle P\rangle}(x)$
$= \Psi_{\langle M\rangle}(x) \uplus \left(\Psi_{\langle N\rangle}(x) \wedge \Psi_{\langle P\rangle}(x)\right)$
$= \Psi_{\langle M+Q\rangle}(x)$ where $\langle Q\rangle = \langle N\rangle \cap \langle P\rangle$ and
$= \langle \Psi_{M+Q}(x)\rangle = \langle \Psi_M(x) + \Psi_Q(x)\rangle$ where
$\Psi_Q(x) = \left\{\mu_N^i(x) \wedge \mu_P^i(x)\middle| i = 1,2,\dots,L(x)\right\}$
Thus, $\langle \Psi_{M+Q}(x)\rangle = \langle \Psi_M(x) + \left\{\mu_N^i(x) \wedge \mu_P^i(x)\middle| i = 1,2,\dots,L(x)\right\}\rangle$
But $\langle \Psi_M(x) + \left\{\mu_N^i(x) \wedge \mu_P^i(x)\middle| i = 1,2,\dots,L(x)\right\}\rangle$

$$= \left\langle \begin{matrix} \langle \Psi_M(x) + \left\{\mu_N^i(x)\middle| i = 1,2,\dots,L(x:N)\right\}\rangle \\ \wedge \langle \Psi_M(x) + \left\{\mu_P^i(x)\middle| i = 1,2,\dots,L(x:P)\right\}\rangle \end{matrix} \right\rangle$$

$= \langle \langle \Psi_M(x) + \Psi_N(x)\rangle \wedge \langle \Psi_M(x) + \Psi_P(x)\rangle\rangle$
$= \Psi_{\langle M+N\rangle}(x) \wedge \Psi_{\langle M+P\rangle}(x)$
$= \Psi_{\langle M\rangle \uplus \langle N\rangle}(x) \wedge \Psi_{\langle M\rangle \uplus \langle P\rangle}(x)$
$= \Psi_{(\langle M\rangle \uplus \langle N\rangle) \cap (\langle M\rangle \uplus \langle P\rangle)}(x)$
In particular,
$\langle M\rangle \uplus (\langle N\rangle \cap \langle P\rangle) = (\langle M\rangle \uplus \langle N\rangle)\cap(\langle M\rangle \uplus \langle P\rangle)$
(x). $\Psi_{\langle M\rangle \uplus \langle N\rangle}(x) = \Psi_{\langle M+N\rangle}(x) = \langle \Psi_M(x) + \Psi_N(x)\rangle$
$= \langle \left(\Psi_M(x) \cup \Psi_N(x)\right) + \left(\Psi_M(x) \cap \Psi_N(x)\right)\rangle$
(Theorem 2.1(ii))
$= \Psi_{\langle M\cup N\rangle \uplus \langle M\cap N\rangle}(x)$ (by definition)
Hence, $\langle M\rangle \uplus \langle N\rangle = \langle M\cup N\rangle \uplus \langle M\cap N\rangle$

Proposition 3.7. For any $M, N, P \in \mathcal{FM}(\mathcal{S})$, we have the following:
(i). $\langle M\rangle \oplus \langle \emptyset\rangle = \langle M\rangle$
(ii). $\langle M\rangle \otimes \langle \emptyset\rangle = \langle \emptyset\rangle$
(iii). $(\langle M\rangle \otimes \langle N\rangle)\otimes \langle P\rangle = \langle M\rangle \otimes (\langle N\rangle \otimes \langle P\rangle)$
(iv). $(\langle M\rangle \oplus \langle N\rangle)\oplus \langle P\rangle = \langle M\rangle \oplus (\langle N\rangle \oplus \langle P\rangle)$

Proof:
(i). $\Psi_{\langle M\rangle \oplus \langle \emptyset\rangle}(x) = \Psi_{\langle M\rangle}(x) + \Psi_{\langle \emptyset\rangle}(x)$
$= \left(\mu_M^i(x) + \mu_\emptyset^i(x) - \mu_M^i(x).\mu_\emptyset^i(x)\middle| i = 1,2,\dots,L(x;M)\right)$
$\left(\mu_M^i(x) + 0 - \mu_M^i(x).0\middle| i = 1,2,\dots,L(x;M)\right)$
$\left(\mu_M^i(x)\middle| i = 1,2,\dots,L(x;M)\right) = \Psi_{\langle M\rangle}(x)$
In particular, $\langle M\rangle \oplus \langle \emptyset\rangle = \langle M\rangle$

(ii). $\Psi_{\langle M\rangle \otimes \langle \emptyset\rangle}(x) = \Psi_{\langle M\rangle}(x).\Psi_{\langle \emptyset\rangle}(x)$
$= \left(\mu_M^i(x).\mu_\emptyset^i(x)\middle| i = 1,2,\dots,L(x;M)\right)$
$= \left(\mu_M^i(x).0\middle| i = 1,2,\dots,L(x;M)\right)$
$= \left(0\middle| i = 1,2,\dots,L(x;M)\right) = \Psi_{\langle \emptyset\rangle}(x)$
In particular, $\langle M\rangle \otimes \langle \emptyset\rangle = \langle \emptyset\rangle$

(iii). $\Psi_{(\langle M\rangle\otimes\langle N\rangle)\otimes\langle P\rangle}(x) = \Psi_{\langle M\rangle\otimes\langle N\rangle}(x).\Psi_{\langle P\rangle}(x)$

$$= \left(\Psi_{\langle M\rangle}(x).\Psi_{\langle N\rangle}(x)\right).\Psi_{\langle P\rangle}(x)$$

$$= \left(\left(\mu_M^i(x).\mu_N^i(x)\right).\mu_P^i(x)\middle| i = 1,2,\dots,max\{L'(x),L(x;P)\}\right)$$

$$= \left(\mu_M^i(x).\left(\mu_N^i(x).\mu_P^i(x)\right)\middle| i = 1,2,\dots,max\{L(x;M),L''(x)\}\right)$$

where $L'(x) = L(x;M,N)$ and $L''(x) = L(x;N,P)$.
Thus,

$$\left(\Psi_{\langle M\rangle}(x).\Psi_{\langle N\rangle}(x)\right).\Psi_{\langle P\rangle}(x)$$
$$= \Psi_{\langle M\rangle}(x).\left(\Psi_{\langle N\rangle}(x).\Psi_{\langle P\rangle}(x)\right)$$

In particular, $\Psi_{(\langle M\rangle\otimes\langle N\rangle)\otimes\langle P\rangle}(x) = \Psi_{\langle M\rangle\otimes(\langle N\rangle\otimes\langle P\rangle)}(x)$ and
$(\langle M\rangle\otimes\langle N\rangle)\otimes\langle P\rangle = \langle M\rangle\otimes(\langle N\rangle\otimes\langle P\rangle)$.

(iv). $\Psi_{(\langle M\rangle\oplus\langle N\rangle)\oplus\langle P\rangle}(x) = \Psi_{\langle M\rangle\oplus\langle N\rangle}(x) + \Psi_{\langle P\rangle}(x)$

$$= \left(\Psi_{\langle M\rangle}(x) + \Psi_{\langle N\rangle}(x)\right) + \Psi_{\langle P\rangle}(x)$$

$$= \begin{pmatrix} \left(\mu_M^i(x) + \mu_N^i(x) - \mu_M^i(x).\mu_N^i(x)\right) + \mu_P^i(x) \\ -\left(\mu_M^i(x) + \mu_N^i(x) - \mu_M^i(x).\mu_N^i(x)\right).\mu_P^i(x) \end{pmatrix}$$

$$= \begin{pmatrix} \mu_M^i(x) + \mu_N^i(x) + \mu_P^i(x) - \mu_M^i(x).\mu_N^i(x) \\ -\mu_M^i(x).\mu_P^i(x) - \mu_N^i(x).\mu_P^i(x) + \mu_M^i(x).\mu_N^i(x).\mu_P^i(x) \end{pmatrix}$$

$$= \begin{pmatrix} \mu_M^i(x) + \left(\mu_N^i(x) + \mu_P^i(x) - \mu_N^i(x).\mu_P^i(x)\right) \\ -\mu_M^i(x)\left(\mu_N^i(x) +. \mu_P^i(x) - \mu_N^i(x).\mu_P^i(x)\right) \end{pmatrix}$$

$$= \Psi_{\langle M\rangle}(x) + \left(\Psi_{\langle N\rangle}(x) + \Psi_{\langle P\rangle}(x)\right)$$
$$= \Psi_{\langle M\rangle}(x) + \Psi_{\langle N\rangle\oplus\langle P\rangle}(x)$$
$$= \Psi_{\langle M\rangle\oplus(\langle N\rangle\oplus\langle P\rangle)}(x).$$

Hence, $\Psi_{(\langle M\rangle\oplus\langle N\rangle)\oplus\langle P\rangle}(x) = \Psi_{\langle M\rangle\oplus(\langle N\rangle\oplus\langle P\rangle)}(x)$
In particular, $(\langle M\rangle\oplus\langle N\rangle)\oplus\langle P\rangle = \langle M\rangle\oplus(\langle N\rangle\oplus\langle P\rangle)$

Proposition 3.8. For any $M,N \in \mathcal{FM}(\mathcal{S})$, the following results are valid:

(i). $(M\cup N)^* = M^*\cup N^*$
(ii) $(M\cap N)^* = M^*\cap N^*$
(iii). $M \subseteq N \to M^* \subseteq N^*$

Proof:
(i).Let $x \in \mathcal{S}|x \in (M\cup N)^*$. Then $\Psi_{M\cup N}(x) \neq \Psi_\emptyset(x)$ (by definition)
But $\Psi_{M\cup N}(x) = \Psi_M(x)\cup\Psi_N(x)$
Thus, $\Psi_M(x)\cup\Psi_N(x) \neq \Psi_\emptyset(x)$
Thus we have $\alpha \in \Psi_M(x)\cup\Psi_N(x)$ such that $\alpha > 0$ and $\alpha \notin \Psi_\emptyset(x)$. In particular, $C_{\Psi_M(x)\cup\Psi_N(x)}(\alpha) > 0$.
i.e. $max\{C_{\Psi_M(x)}(\alpha), C_{\Psi_N(x)}(\alpha)\} > 0$
Thus either $C_{\Psi_M(x)}(\alpha) > 0$ or $C_{\Psi_N(x)}(\alpha) > 0$

In this case, $\Psi_M(x) \neq \Psi_\emptyset(x)$ or $\Psi_N(x) \neq \Psi_\emptyset(x)$.
i.e $x \in M^* \cup N^*$.
In particular, $(M \cup N)^* \subseteq M^* \cup N^*$ (1)
However, if $y \in \mathcal{S} | y \in M^* \cup N^*$, then $y \in M^*$ or $y \in N^*$.
In this case, $\Psi_M(y) \neq \Psi_\emptyset(y)$ or $\Psi_N(y) \neq \Psi_\emptyset(y)$ (by definition).
In particular, $\Psi_M(y) \cup \Psi_N(y) \neq \Psi_\emptyset(y)$.
i.e $\Psi_{M \cup N}(y) \neq \Psi_\emptyset(x)$ and we have $y \in (M \cup N)^*$.
Thus, $M^* \cup N^* \subseteq (M \cup N)^*$ (2)
Hence, the result follows from (1) and (2) above

(ii). Let $x \in \mathcal{S} | x \in (M \cap N)^*$. Then $\Psi_{M \cap N}(x) \neq \Psi_\emptyset(x)$
(by definition).
But $\Psi_{M \cap N}(x) = \Psi_M(x) \cap \Psi_N(x)$
Thus we have $\alpha \in \Psi_M(x) \cap \Psi_N(x)$ such that $\alpha > 0$ and $\alpha \notin \Psi_\emptyset(x)$.
In particular, $C_{\Psi_M(x) \cap \Psi_N(x)}(\alpha) > 0$.
i.e. $min\{C_{\Psi_M(x)}(\alpha), C_{\Psi_N(x)}(\alpha)\} > 0$
Thus $C_{\Psi_M(x)}(\alpha) > 0$ and $C_{\Psi_N(x)}(\alpha) > 0$
In this case, $\Psi_M(x) \neq \Psi_\emptyset(x)$ and $\Psi_N(x) \neq \Psi_\emptyset(x)$.
I.e $x \in M^* \cap N^*$
In particular, $(M \cap N)^* \subseteq M^* \cap N^*$ (3)
However, if $y \in \mathcal{S} | y \in M^* \cap N^*$, then $y \in M^*$ and $y \in N^*$.
In this case, $\Psi_M(y) \neq \Psi_\emptyset(y)$ and $\Psi_N(y) \neq \Psi_\emptyset(y)$ (by definition).
In particular, $\Psi_M(y) \cap \Psi_N(y) \neq \Psi_\emptyset(y)$.
i.e $\Psi_{M \cap N}(y) \neq \Psi_\emptyset(y)$
and we have $y \in (M \cap N)^*$

Thus, $M^* \cap N^* \subseteq (M \cap N)^*$ (4)
Hence, the result follows from (3) and (4) above.

(ii). $M \subseteq N \leftrightarrow \Psi_M(x) \subseteq \Psi_N(x)$ for all $x \in \mathcal{S}$.
Now let $x \in M^*$. We have $\Psi_M(x) \neq \Psi_\emptyset(x)$
(by definition).
Thus we have $\alpha \in \Psi_M(x)$ such that $\alpha > 0$. i.e. $C_{\Psi_M(x)}(\alpha) > 0$. But $\Psi_M(x) \subseteq \Psi_N(x)$.
Thus $0 < C_{\Psi_M(x)}(\alpha) < C_{\Psi_N(x)}(\alpha)$ and $C_{\Psi_N(x)}(\alpha) > 0$.
Hence, $\Psi_N(x) \neq \Psi_\emptyset(x)$ and $x \in N^*$.
In particular, $M \subseteq N \rightarrow M^* \subseteq N^*$

Corollary 3.9. If $M_i \in \mathcal{FM}(\mathcal{S}), i = 1,2, \dots, n$, then
(i). $(\bigcap_{i=1}^n M_i)^* = \bigcap_{i=1}^n M_i^*$ and
(ii). $(\bigcup_{i=1}^n M_i)^* = \bigcup_{i=1}^n M_i^*$

Proposition 3.10. For any $M, N \in \mathcal{FM}(\mathcal{S})$, the following results are valid:
(i). $(\langle M \rangle \cup \langle N \rangle)^* = \langle M \rangle^* \cup \langle N \rangle^*$
(ii). $(\langle M \rangle \cap \langle N \rangle)^* = \langle M \rangle^* \cap \langle N \rangle^*$
(iii). $\langle M \rangle \subseteq \langle N \rangle \rightarrow \langle M \rangle^* \subseteq \langle N \rangle^*$

Proof:

(i). Let $x \in (\langle M\rangle \cup \langle N\rangle)^*$. We show that $x \in \langle M\rangle^* \cup \langle N\rangle^*$.
Now, $x \in (\langle M\rangle \cup \langle N\rangle)^* \rightarrow \Psi_{\langle M\rangle \cup \langle N\rangle}(x) \succ \Psi_{\langle \emptyset\rangle}(x)$
(by definition).
But $\Psi_{\langle M\rangle \cup \langle N\rangle}(x) = \Psi_{\langle M\rangle}(x) \vee \Psi_{\langle N\rangle}(x)$ and

$$\Psi_{\langle M\rangle}(x) \vee \Psi_{\langle N\rangle}(x) = \left(\mu_M^i(x) \vee \mu_N^i(x) \middle| \, i = 1,2,\dots,L(x)\right)$$

Thus, $\mu_M^i(x) \vee \mu_N^i(x) > 0$ for some $i \in [1, L(x)]$
In particular, either $\mu_M^i(x) > 0$ or $\mu_N^i(x) > 0$ for such i.
Hence, $\Psi_{\langle M\rangle}(x) \succ \Psi_{\langle \emptyset\rangle}(x)$ or $\Psi_{\langle N\rangle}(x) \succ \Psi_{\langle \emptyset\rangle}(x)$.
i.e $x \in \langle M\rangle^* \cup \langle N\rangle^*$.
$(\langle M\rangle \cup \langle N\rangle)^* \subseteq \langle M\rangle^* \cup \langle N\rangle^*$ (1)
However, if $y \in \langle M\rangle^* \cup \langle N\rangle^*$, then either $y \in \langle M\rangle^*$ or $y \in \langle N\rangle^*$.
In this case, $\mu_M^i(y) > 0$ or $\mu_N^i(y) > 0$ for some $i \in [1, L(x)]$.
In particular, $\mu_M^i(y) \vee \mu_N^i(y) > 0$ for such i.
Thus, $\Psi_{\langle M\rangle \cup \langle N\rangle}(y) \succ \Psi_{\langle \emptyset\rangle}(y)$. i.e $y \in (\langle M\rangle \cup \langle N\rangle)^*$ and $\langle M\rangle^* \cup \langle N\rangle^* \subseteq (\langle M\rangle \cup \langle N\rangle)^*$ (2)
The result follows from (1) and (2) above.

(ii) Let $x \in (\langle M\rangle \cap \langle N\rangle)^*$. Thus, $\Psi_{\langle M\rangle \cap \langle N\rangle}(x) \succ \Psi_{\langle \emptyset\rangle}(x)$
(by definition).
i.e $\mu_M^i(x) \wedge \mu_N^i(x) > 0$ for some $i \in [1, L(x)]$.
Hence, $\mu_M^i(x) > 0$ and $\mu_N^i(x) > 0$ for such i.
In particular, $\Psi_{\langle M\rangle}(x) \succ \Psi_{\langle \emptyset\rangle}(x)$ and $\Psi_{\langle N\rangle}(x) \succ \Psi_{\langle \emptyset\rangle}(x)$.
i.e $x \in \langle M\rangle^* \cap \langle N\rangle^*$.
Thus, $(\langle M\rangle \cap \langle N\rangle)^* \subseteq \langle M\rangle^* \cap \langle N\rangle^*$ (1)
However, if $y \in \langle M\rangle^* \cap \langle N\rangle^*$.
We have $y \in \langle M\rangle^*$ and $y \in \langle N\rangle^*$. i.e $\Psi_{\langle M\rangle}(y) \succ \Psi_{\langle \emptyset\rangle}(y)$ and $\Psi_{\langle N\rangle}(y) \succ \Psi_{\langle \emptyset\rangle}(y)$.
In this case, $\mu_M^i(y) > 0$ for some $i \in [1, L(y:M)]$ and $\mu_N^j(y) > 0$ for some $j \in [1, L(y:N)]$.
If $L(y) = L(y:M)$, then $i \leq L(y:M)$ and $j \leq L(y:M)$.
Now for $i = j$, we have $\mu_M^i(y) \wedge \mu_N^i(y) > 0$ and
$\Psi_{\langle M\rangle \cap \langle N\rangle}(y) \succ \Psi_{\langle \emptyset\rangle}(y)$.
Thus, $y \in (\langle M\rangle \cap \langle N\rangle)^*$ and $\langle M\rangle^* \cap \langle N\rangle^* \subseteq (\langle M\rangle \cap \langle N\rangle)^*$ (2)'
If $j < i$, then $\mu_M^j(y) > 0$ and $\mu_M^j(y) \wedge \mu_N^j(y) > 0$ and
$\Psi_{\langle M\rangle \cap \langle N\rangle}(y) \succ \Psi_{\langle \emptyset\rangle}(y)$. Thus, $y \in (\langle M\rangle \cap \langle N\rangle)^*$ and
$\langle M\rangle^* \cap \langle N\rangle^* \subseteq (\langle M\rangle \cap \langle N\rangle)^*$ (2)''
If $i < j$, we have $\mu_M^j(y) > 0$ and $\mu_M^j(y) \wedge \mu_N^j(y) > 0$.
Thus, $\Psi_{\langle M\rangle \cap \langle N\rangle}(y) \succ \Psi_{\langle \emptyset\rangle}(y)$ and $y \in (\langle M\rangle \cap \langle N\rangle)^*$
In particular, $\langle M\rangle^* \cap \langle N\rangle^* \subseteq (\langle M\rangle \cap \langle N\rangle)^*$ (2)'''
Similar arguments can be advanced for $L(y) = L(y:N)$ to get the same results as in (2)'-(2)''' above. Thus,
$\langle M\rangle^* \cap \langle N\rangle^* \subseteq (\langle M\rangle \cap \langle N\rangle)^*$ (3)
Hence, from (1) and (3) above, we have
$(\langle M\rangle \cap \langle N\rangle)^* = \langle M\rangle^* \cap \langle N\rangle^*$.
(iii). Suppose $\langle M\rangle \subseteq \langle N\rangle$, we have $\Psi_{\langle M\rangle}(x) \preccurlyeq \Psi_{\langle N\rangle}(x)$.
In particular, $\mu_M^i(x) \leq \mu_N^i(x)$, $i = 1,2,\dots,L(x)$.

Thus, if $y \in \langle M \rangle^*$, we have $\Psi_{\langle M \rangle}(y) \succ \Psi_{\langle \emptyset \rangle}(y)$
In particular, $\Psi_{\langle \emptyset \rangle}(y) \prec \Psi_{\langle M \rangle}(y) \preccurlyeq \Psi_{\langle N \rangle}(y)$ and
$\Psi_{\langle \emptyset \rangle}(y) \prec \Psi_{\langle N \rangle}(y)$. Hence, $y \in \langle N \rangle^*$. Thus, $\langle M \rangle^* \subseteq \langle N \rangle^*$.
In particular,$\langle M \rangle \subseteq \langle N \rangle \to \langle M \rangle^* \subseteq \langle N \rangle^*$.

Corollary 3.11. If $M_i \in \mathcal{FM}(\mathcal{S}), i = 1,2, \dots, n$, then
(i). $(\bigcap_{i=1}^{n} \langle M \rangle_i)^* = \bigcap_{i=1}^{n} \langle M \rangle_i^*$ and
(ii). $(\bigcup_{i=1}^{n} \langle M \rangle_i)^* = \bigcup_{i=1}^{n} \langle M \rangle_i^*$

Proposition 3.12. For any $M, N \in \mathcal{FM}(\mathcal{S})$, the following results are valid:
(i). $\langle M \rangle^* = M^*$
(ii) $(\langle M \rangle \cup \langle N \rangle)^* = M^* \cup N^*$
(iii). $(\langle M \rangle \cap \langle N \rangle)^* = M^* \cap N^*$

Proof:
(i). If $x \in \langle M \rangle^*$, then $\Psi_{\langle M \rangle}(x) \succ \Psi_{\langle \emptyset \rangle}(x)$.

In particular, $\mu_M^i(x) > 0$ for some $i \in [1, L(x{:}\,M)]$.

Thus, $\Psi_M(x) \neq \emptyset$ and $x \in M^*$. In particular,
$\langle M \rangle^* \subseteq M^*$ (1)

If $y \in M^*$, then $\Psi_M(y) \neq \Psi_\emptyset(y)$ (by definition)

Hence, $\mu_M^i(y) > 0$ for some $i \in [1, L(x{:}\,M)]$ and
$\Psi_{\langle M \rangle}(y) \succ \Psi_{\langle \emptyset \rangle}(y)$. In particular,
$M^* \subseteq \langle M \rangle^*$ (2)

Hence from (1) and (2) above, the result follows.

(ii)-(iii). The results can be easily seen using (i) above with reference to proposition 3.10 (i-ii)

Corollary 3.13. If $M_i \in \mathcal{FM}(\mathcal{S}), i = 1,2, \dots, n$, then
(i). $(\bigcup_{i=1}^{n} \langle M \rangle_i)^* = \bigcup_{i=1}^{n} M_i^*$
(ii). $(\bigcap_{i=1}^{n} \langle M \rangle_i)^* = \bigcap_{i=1}^{n} M_i^*$ and

Proof:
(i-ii). The results from follows from corollary 3.11 and proposition 3.12

Definition 3.3 [12]. For any fuzzy mset $M \in \mathcal{FM}(\mathcal{S})$, the cardinality of $\langle M \rangle$ denoted $|\langle M \rangle|$ is defined :
$|\langle M \rangle| = \sum_{x \in \mathcal{S}} \sum_{j=1}^{L(x;M)} \mu_M^j(x)$ or
$|\langle M \rangle| = \sum_{x \in \mathcal{S}} |\langle M \rangle|_x$ where $|\langle M \rangle|_x = \sum_{j=1}^{L(x;M)} \mu_M^j(x)$

Proposition 3.14 (principle of Inclusion/exclusion). For any $M, N \in \mathcal{FM}(\mathcal{S})$,
$|\langle M \rangle \cup \langle N \rangle| = |\langle M \rangle| + |\langle N \rangle| - |\langle M \rangle \cap \langle N \rangle|$

Proof:

$$|\langle M\rangle \cup \langle N\rangle|_x = \sum_{i=1}^{L(x)} \mu_{M\cup N}^i(x) = \sum_{i=1}^{L(x)} \mu_M^i(x) \vee \mu_N^i(x)$$

But
$\mu_M^i(x) + \mu_N^i(x) = \left(\mu_M^i(x) \vee \mu_N^i(x)\right) + \left(\mu_M^i(x) \wedge \mu_N^i(x)\right)$

Thus, $\left(\mu_M^i(x) \vee \mu_N^i(x)\right) = \left(\mu_M^i(x) + \mu_N^i(x)\right) - \left(\mu_M^i(x) \wedge \mu_N^i(x)\right)$
Hence, $\sum_{i=1}^{L(x)} \mu_M^i(x) \vee \mu_N^i(x) = \sum_{i=1}^{L(x)} \left(\left(\mu_M^i(x) + \mu_N^i(x)\right) - \left(\mu_M^i(x) \wedge \mu_N^i(x)\right)\right)$
$= \sum_{i=1}^{L(x)} \left(\mu_M^i(x) + \mu_N^i(x)\right) - \sum_{i=1}^{L(x)}\left(\mu_M^i(x) \wedge \mu_N^i(x)\right)$

$= \sum_{i=1}^{L(x)} \mu_M^i(x) + \sum_{i=1}^{L(x)} \mu_N^i(x) - \sum_{i=1}^{L(x)}\left(\mu_M^i(x) \wedge \mu_N^i(x)\right)$

$$= \sum_{i=1}^{L(x;M)} \mu_M^i(x) + \sum_{i=1}^{L(x;N)} \mu_N^i(x) - \sum_{i=1}^{L(x)} \left(\mu_M^i(x) \wedge \mu_N^i(x)\right)$$

$= |\langle M\rangle|_x + |\langle N\rangle|_x - |\langle M\rangle \cap \langle N\rangle|_x$

Thus, $|\langle M\rangle \cup \langle N\rangle|_x = |\langle M\rangle|_x + |\langle N\rangle|_x - |\langle M\rangle \cap \langle N\rangle|_x$

In particular, $\sum_{x\in\mathcal{S}} |\langle M\rangle \cup \langle N\rangle|_x = \sum_{x\in\mathcal{S}}(|\langle M\rangle|_x + |\langle N\rangle|_x - |\langle M\rangle \cap \langle N\rangle|_x)$

$= \sum_{x\in\mathcal{S}}|\langle M\rangle|_x + \sum_{x\in\mathcal{S}}|\langle N\rangle|_x - \sum_{x\in\mathcal{S}}|\langle M\rangle \cap \langle N\rangle|_x$
i.e $\sum_{x\in\mathcal{S}} |\langle M\rangle \cup \langle N\rangle|_x = \sum_{x\in\mathcal{S}}|\langle M\rangle|_x + \sum_{x\in\mathcal{S}}|\langle N\rangle|_x - \sum_{x\in\mathcal{S}}|\langle M\rangle \cap \langle N\rangle|_x$

In particular, $|\langle M\rangle \cup \langle N\rangle| = |\langle M\rangle| + |\langle N\rangle| - |\langle M\rangle \cap \langle N\rangle|$

Proposition 3.15. For any $M, N \in \mathcal{FM}(\mathcal{S})$,
(i). $(\langle M\rangle^c)^c = \langle M\rangle$
(ii). $(\langle M\rangle \cup \langle N\rangle)^c = \langle M\rangle^c \cap \langle N\rangle^c$
(iii). $(\langle M\rangle \cap \langle N\rangle)^c = \langle M\rangle^c \cup \langle N\rangle^c$

Proof:
(i). $\Psi_{\langle M\rangle^c}(x) = \left(\mu_{M'}^i(x) \middle| i = 1,2,\dots,L(x;M)\right)$ where
$1 - \mu_M^i(x) = \mu_{M'}^i(x)$ (by definition).

Now $\Psi_{(\langle M\rangle^c)^c}(x) = \left(\mu_{M''}^i(x) \middle| i = 1,2,\dots,L(x;M)\right)$

where $\mu_{M''}^i(x) = 1 - \left(1 - \mu_M^i(x)\right) = \mu_M^i(x)$.

Thus, $\Psi_{(\langle M\rangle^c)^c}(x) = \left(\mu_M^i(x). \middle| i = 1,2,\dots,L(x;M)\right)$
$= \Psi_{\langle M\rangle}(x)$.

Hence, $(\langle M\rangle^c)^c = \langle M\rangle$.

(ii). $\Psi_{(\langle M\rangle\cup\langle N\rangle)^c}(x) = \left(1 - \left(\mu_M^i(x) \vee \mu_N^i(x)\right) \middle| i = 1,2,\dots,L(x)\right)$

$= \left(\left(1 - \mu_M^i(x)\right) \wedge \left(1 - \mu_N^i(x)\right) \middle| i = 1,2,\dots,L(x)\right)$

$= \left(\mu_{M'}^i(x) \wedge \mu_{N'}^i(x \middle| i = 1,2,\dots,L(x)\right)$

$= \Psi_{\langle M\rangle^c\cap\langle N\rangle^c}(x).$

In particular, $(\langle M\rangle\cup\langle N\rangle)^c = \langle M\rangle^c\cap\langle N\rangle^c$

(iii). $\Psi_{(\langle M\rangle\cap\langle N\rangle)^c}(x) = \left(1 - \left(\mu_M^i(x) \wedge \mu_N^i(x)\right) \middle| i = 1,2,\dots,L(x)\right)$

$= \left(\left(1 - \mu_M^i(x)\right) \vee \left(1 - \mu_N^i(x)\right) \middle| i = 1,2,\dots,L(x)\right)$

$= \left(\mu_{M'}^i(x) \vee \mu_{N'}^i(x \middle| i = 1,2,\dots,L(x)\right)$

$= \Psi_{\langle M\rangle^c\cup\langle N\rangle^c}(x)$

In particular, $(\langle M\rangle\cap\langle N\rangle)^c = \langle M\rangle^c\cup\langle N\rangle^c$

4 Conclusion

In this paper, a symbolic representation of a fuzzy membership sequenced mset has been introduced and the additive union and complement operations in the literature also redefined. The study of algebraic properties such as commutativity, identity, idempotent, associativity, distributivity, absorption, Demorgan's laws and the principle of Inclusion/exclusion have been presented and proved valid in the context of these structures. The compatibility of the union and intersection operations on sets and these structures via their root sets also established. It is hopeful that the closure of the union and intersection operations established on power-whole and powerfull fuzzy membership sequenced msets will pave way for future research on these structures as topological space.

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Competing Interests

Author has declared that no competing interests exist.

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