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On Nörlund summability of double Fourier series

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Abstract: In this research paper, the authors studied some problems related to harmonic summability of double Fourier series on Nörlund summability method. These results constitute substantial extension and generalization of related work of Moricz [1] and Rhodes *et al.*, [2]. We also constructed a new result on $(N, p_b^{(1)}, p_a^{(2)})$ by regular Nörlund method of summability.

Keywords: Nörlund sumambility; Double Fourier series; Double matrix summability.

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1. Introduction

Let $f(\alpha,\beta)$ be Lebesgue integral in the square $R(-\pi,\pi;-\pi,\pi)$ and be of period 2π in each of the variables α and β. Then the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\gamma_{n,m}\left\{r_{m,n}\cos m\alpha.\cos n\beta+s_{mn}\ \sin m\alpha.\cos n\beta+t_{mn}\ \cos m\alpha.\sin n\beta+q_{mn}\sin m\alpha.\sin n\beta\right\} \tag{1}$$

is called the double Fourier series associated with the function $f(\alpha,\beta)$; (see [2,3]) where

$$\gamma_{mn}=\begin{cases}\frac{1}{4}, & \text{for } m=0,n=0;\\ \frac{1}{2}, & \text{for } m=0,n>0 \text{ or } m>0,n=0;\\ 1, & \text{for } m,n>0,\end{cases} \tag{2}$$

$$r_{mm}=\frac{1}{\pi^2}\int\int_R f(\alpha,\beta) \tag{3}$$

$$cosm\alpha.\cos n\beta\ d\alpha d\beta, \tag{4}$$

$$S_{mn}=\frac{1}{\pi^2}\int\int_R f(\alpha,\beta).\sin m\alpha.\cos n\beta\ d\alpha d\beta, \tag{5}$$

$$t_{mn}=\frac{1}{\pi^2}\int\int_R f(\alpha,\beta)\cos m\alpha.\sin n\beta\ d\alpha d\beta, \tag{6}$$

$$q_{mn}=\frac{1}{\pi^2}=\int\int_R f(\alpha,\beta)\sin m\alpha.\sin n\beta\ d\alpha d\beta. \tag{7}$$

Also, we have

$$\chi(\alpha,\beta)=\chi_{x,y}(\alpha,\beta)=\frac{1}{4}\left\{f(x+\alpha,y+\beta)+f(x-\alpha,y+\beta)+f(x+\alpha,y-\beta)+f(x-\alpha,y-\beta)-4f(\alpha,\beta)\right\}. \tag{8}$$

Definition 1. [4,5] Let$\{p_m^{(1)}\}$ and $\{p_n^{(2)}\}$ are two sequence of constants, real or complex. Let

$$\begin{cases} P_m^{(1)} = p_0^{(1)} + p_1^{(1)} + p_2^{(1)} + ... + p_m^{(1)}, \\ P_n^{(2)} = p_0^{(2)} + p_1^{(2)} + p_2^{(2)} + ... + p_n^{(2)}. \end{cases} \tag{9}$$

We shall also consider a double Nörlund transform of $\{a_{mn}\}$. Then the double Nörlund transform is

$$V_{mn} = \frac{1}{P_m^{(1)} P_n^{(2)}} . \sum_{l=0}^{m} \sum_{g=0}^{n} p_{m-l}^{(1)} p_{g-n}^{(2)} a_{lg}. \tag{10}$$

Definition 2. [4,5] The double sequence $\{a_{lg}\}$ is said to be Nörlund summable to a limit v if

$$V_{mn} \to V, (m, n) \to (\infty, \infty). \tag{11}$$

It is also known as summable $(N, p_m^{(1)} p_n^{(2)})$.

Definition 3. [1,2,4–6] If

$$\begin{cases} p_m^{(1)} = 1, & \text{for } m = 0, 1, 2, 3, ...; \\ p_n^{(2)} = 1, & \text{for } m = 0, 1, 2, ..., \end{cases} \tag{12}$$

then the double Nörlund transform reduces to double Cesàro transform of order one. This summability method is known as Cesàro summability $(C, 1, 1)$.

Definition 4. [1,2,5,6] If

$$p_m^{(1)} = \frac{1}{m+1}; m = 0, 1, 2, ... \text{ and } p_n^{(2)} = \frac{1}{n+1}, n = 0, 1, 2, ...,$$

then the double Nörlund summability $(N, p_m^{(1)}, p_n^{(2)})$ becomes Harmonic summability and is denoted by $(H, 1, 1)$.

Definition 5. [5] If for any $\gamma \geq 1, V_{mn} \to V, (m, n) \to (-\infty, \infty)$ in such a manner that $\gamma \geq \frac{m}{n}, \gamma \geq \frac{n}{m}$, then the sequence $\{a_{lg}\}$ is said to be restrictedly summable N_p at (x, y) to the same limit.

2. Known results

In 1953, Chow [7], for the first time studied Cesáro summability of double Fourier series. In 1958, Sharma [3] extended the results of Chow for $(H, 1, 1)$ summability which is weaker than $(C, 1, 1)$ summability of double Fourier series. There are several results on Nörlund summability of Fourier series [1–10]. This motivates us to study on the Nörlund summability of Fourier series in more generalized as particular cases. Therefore, in an attempt to make an advance in this research work, we study the double Fourier series and its conjugate series by Nörlund method. In 1963, Sing [11] proved the following theorem;

Theorem 1. *If*

$$\int_0^V \left|\chi(y)\right| dy = O\left(\frac{v}{\log v^{-1}}\right), \tag{13}$$

where $\chi(y)$= f(v+y)+f(v-y)-2f(y) as $v \to 0$, then the Fourier series of $f(u)$ at v=y is summable (N, p_n) to $f(y)$ where $\{p_n\}$ is a real non-negative, non-increasing sequence such that

$$\sum_{a=2}^{n} \left(\frac{p_a}{a. \log a}\right) = O\left(P_n\right). \tag{14}$$

Dealing with this topics, Pati [12] proved the following theorem;

Theorem 2. *If (N, p_n) be a regular Nörlund methods defined by a positive, monotonic decreasing sequences of coefficients $\{p_n\}$ such that*

$$\log n = O(P_n), P_n \to \infty, as\ n \to \infty, \tag{15}$$

then if

$$\zeta(x) = \int_0^z |\chi(y)|\, dy = \left[\frac{z}{P_{(\frac{1}{z})}}\right] \tag{16}$$

as $z \to +0$, then the given Fourier series $\chi(z)$ of $z = y$ is summable (N, p_n) to $\chi(y)$.

3. Main Results

In this present research paper, we have established the following theorems which are the extended form of [2,11].

Theorem 3. *If $(\alpha, \beta) \to (0, 0)$,*

$$\int_0^\alpha \int_0^\beta |\chi(s,t)|\, dsdt = O\left(\frac{\alpha}{\log \alpha^{-1}} . \frac{\beta}{\log \beta^{-1}}\right) \tag{17}$$

$$\int_\delta^\pi ds \int_0^\beta |\chi(s,t)|\, dt = O\left(\frac{\beta}{\log \beta^{-1}}\right), (0 < \delta < \pi) \tag{18}$$

and

$$\int_\delta^\pi dt \int_0^\alpha |\chi(s,t)|\, ds = O\left(\frac{\alpha}{\log \alpha^1}\right), (0 < \delta < \pi), \tag{19}$$

then the double Fourier series of $f(\alpha, \beta)$ at $\alpha = x$ and $\beta = y$ is summable $\left(N, p_m^{(1)} p_n^{(2)}\right)$ to the sum $f(x, y)$ where $\{p_n^{(v)}\}$, $(v = 1, 2)$ are real non-negative, non-increasing sequence of constants such that

$$\sum_{k=2}^{n} \left(\frac{p_k^{(v)}}{k. \log k}\right) = O\left(P_n^{(v)}\right), (v = l, 2). \tag{20}$$

The objective of this present work is to generalize the theorem B to a more general class of Nörlund summability for double Fourier series. We prove the following theorem:

Theorem 4. *Let us suppose that the sequence $\{p_b^{(1)}\}$ and $\{p_a^{(2)}\}$ are positively and monotonically decreasing sequence of constants such that*

$$P_b^{(1)} = \sum_{c=0}^{b} P_c^{(1)} \quad and \quad P_a^{(2)} = \sum_{c=0}^{a} P_c^{(2)}, \tag{21}$$

$P_b^{(1)} \to \infty$, $P_a^{(2)} \infty$ and if

$$\chi(\alpha, \beta) = \int_0^\alpha ds \int_0^\beta |\chi(\alpha t) dt| = O\left[\frac{p^{(1)}_{(\frac{1}{\alpha})}}{P^{(1)}_{(\frac{1}{\alpha})}} . \frac{p^{(2)}_{(\frac{1}{\beta})}}{P^{(2)}_{(\frac{1}{\beta})}}\right], \tag{22}$$

then double Fourier series (1) of the function $f(x, y)$ is summable to the sum S at $\alpha = m$ and $\beta = n$, when $(N, p_b^{(1)}, p_a^{(2)})$ is regular Nörlund method.

The following lemmas are required in the proof of our theorems;

Lemma 1. *If $\{p_n\}$ is non-negative and non-increasing, then for $0 \le a \le b \le \infty, 0 \le t \le \pi$ and for any n, we have*

$$\left|\sum_{k=a}^{b} p_k e^{i(n-k)t}\right| \le AP_{[t^{-1}]}. \tag{23}$$

Lemma 2. *Under the condition of Lemma 1,*

$$\left|\sum_{k=0}^{n} \frac{p_k sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = O\left(nP_n\right), \qquad 0 \le t \le \frac{1}{n}. \tag{24}$$

Lemma 3. *Under the condition of Lemma 1,*

$$\left|\sum_{k=0}^{n} p_k \frac{sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = O\left[\frac{1}{t}.P_{[t^{-1}]}\right], \qquad for \qquad \frac{1}{n} \le t \le \delta. \tag{25}$$

Lemma 4. *Under the condition of Lemma 1,*

$$\left|\sum_{k=0}^{n} p_k \frac{sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = O(1), \qquad for \qquad 0 \le \delta < t \le \pi. \tag{26}$$

they are uniformly in each of the intervals.

Proof of Theorem 3. Let $U_{mn}(x,y;f) = U_{mn}$ denotes the rectangular $(m,n)^{th}$ partial sum of the series (1), then we must have

$$U_{mn}(x,y;f) - f(x,y) = \frac{1}{\pi^2}\int_0^T\int_0^\pi \chi(\alpha,\beta)D_m^1(\alpha)D_n^2(\beta)d\alpha d\beta, \tag{27}$$

where

$$D_m^1(\alpha) = \frac{\sin\left(m+\frac{1}{2}\right)\alpha}{2\sin\frac{\alpha}{2}}, \tag{28}$$

and

$$D_n^2(\beta) = \frac{\sin\left(n+\frac{1}{2}\right)\beta}{2\sin\frac{\beta}{2}}, \tag{29}$$

where $D_m^1(\alpha)$ and $D_n^2(\beta)$ are respectively denote the Dirichlet kernels.

Let $\left\{V_{mn}(x,y)\right\}$ denote the double Nörlund transform of the sequence $\left\{V_{mn} - f(x,y)\right\}$, then

$$\begin{aligned}
V_{mn}(x,y) &= \frac{1}{p_m^{(1)}.p_n^{(2)}}.\sum_{l=0}^{m}\sum_{g=0}^{n} p_{m-l}^{(1)}\, p_{n-g}^{(2)}\left\{U_{lg} - f(x,y)\right\}\\
&= \frac{1}{p_m^{(1)}p_n^{(2)}}\sum_{l=0}^{m}\sum_{g=0}^{n}\left\{p_{m-l}^{(1)}p_{n-g}^{(2)}.\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \chi(\alpha,\beta).D_l^1(\alpha)D_g^2(\beta)d\alpha d\beta\right\}\\
&= \int_0^\pi\int_0^\pi \chi(\alpha,\beta)\left\{\frac{1}{2\pi P_m^{(1)}}.\sum_{l=0}^{m} p_{m-l}^{(1)}\frac{\sin\left(l+\frac{1}{2}\right)l}{\sin\frac{\alpha}{2}}\right\}.\left\{\frac{1}{2\pi p_n^{(2)}}\sum_{g=0}^{n} p_{g-n}^{(2)}.\frac{\sin\left(g+\frac{1}{2}\right)\beta}{\sin\frac{\beta}{2}}\right\}d\alpha.d\beta\\
&= \int_0^\pi\int_0^\pi \chi(\alpha,\beta).N_m^{(1)}(\alpha).N_n^{(2)}(\beta)d\alpha.d\beta,
\end{aligned} \tag{30}$$

where

$$N_m^{(1)}(\alpha)=\frac{1}{2\pi P_m^{(1)}}.\sum_{l=0}^{m}p_{m-l}^{(1)}\frac{\sin\left(m-l+\frac{1}{2}\right)\alpha}{\sin\frac{\alpha}{2}},\tag{31}$$

and

$$N_n^{(2)}(\beta)=\frac{1}{2\pi P_n^{(2)}}\sum_{g=0}^{n}p_{g-n}^{(2)}.\frac{\sin\left(n-g+\frac{1}{2}\right)\beta}{\sin\frac{\beta}{2}}.\tag{32}$$

Also Eq. (30) can be written as

$$\begin{aligned}U_{mn}(x,y)-f(x,y)&=\int_0^{\pi}\int_0^{\pi}\chi(\alpha,\beta)N_m^{(1)}(\alpha).N_n^{(2)}(\beta)d\alpha.d\beta\\&=\left(\int_0^{\pi}\int_0^{\tau}+\int_0^{\delta}\int_{\tau}^{\pi}+\int_{\delta}^{\pi}\int_0^{\tau}+\int_{\delta}^{\pi}\int_{\tau}^{\pi}\right)\chi(\alpha,\beta).N_m^{(1)}(\alpha).N_n^{(2)}(\beta)d\alpha.d\beta\\&=I_1+I_2+I_3+I_4,\quad(\text{say}).\end{aligned}\tag{33}$$

By hypothesis and using the results of (24) and (25), we easily obtain

$$\begin{aligned}|I_4|&=\left|\int_{\delta}^{\pi}\int_{\tau}^{\pi}\chi(\alpha,\beta).N_m^{(1)}(\alpha).N_n^{(2)}(\beta)\right|\\&=O\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\delta}^{\pi}\int_{\tau}^{\pi}\left|\chi(\alpha,\beta)N_m^{(1)}(\alpha).N_n^{(2)}(\beta)\right|d\alpha.d\beta\right)\\&=\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_0^{\pi}\int_0^{\pi}\left|\chi(\alpha,\beta)\right|d\alpha.d\beta\right)\qquad\left(\text{As }\ N_n^{(2)}(\beta),\ \text{ and }\ N_m^{(1)}(\alpha)\ \text{ are even function}\right)\\&=O(1).\end{aligned}\tag{34}$$

Also, for I_3, we have

$$\begin{aligned}I_3&=\int_{\delta}^{\pi}N_m^{(1)}(\alpha)d\alpha\int_0^{\tau}\chi(\alpha,\beta)N_n^{(2)}(\beta)d\beta\\&=\int_{\delta}^{\pi}N_m^{(1)}(\alpha)d\alpha\left\{\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\delta}\right\}\chi(\alpha,\beta).N_n^{(2)}(\beta)d\beta\\&=I_{3,1}+I_{3,2}\quad(\text{say}).\end{aligned}\tag{35}$$

Thus,

$$|I_{3,1}|=O\left(\frac{n}{P_m^{(1)}}\int_0^{\pi}\int_0^{\frac{1}{n}}\left|\chi(\alpha,\beta)\right|d\beta\right)=O\left(\frac{n}{P_m}\right).O\left(\frac{\frac{1}{n}}{\log n}\right)=O(1).\tag{36}$$

Again, by (24) and (25) and hypothesis,

$$\begin{aligned}\left|I_{3,2}\right|&=O\left(\frac{1}{p_m^{(1)}}\int_0^{\pi}d\alpha\int_{\frac{1}{n}}^{\delta}\left|\chi(\alpha,\beta)\right|\frac{1}{P_n^{(2)}}\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}d\beta\right)\\&=O\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\delta}^{\pi}d\alpha\left\{\frac{P_{[\beta^{-1}]}}{\beta}.\chi_1(\alpha,\beta)\right\}_{\frac{1}{n}}^{\delta}-\int_{\frac{1}{n}}^{\delta}\chi_1(\alpha,\beta).d\left(\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right)\right)\\&=(|I_{3,2,1}|)+O\left(|I_{3,2,2}|\right),\quad(\text{say}).\end{aligned}\tag{37}$$

where

$$\chi_1(\alpha,\beta)=\int_0^{\beta}|\chi(\alpha\beta)|dw,\tag{38}$$

and $I_{3,2,1}$ stands for two inner intervals. Thus,

$$
\begin{aligned}
|I_{3,2,1}| &= O\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi d\alpha\left\{\frac{P^{(2)}_{[\tau^{-1}]}}{\tau}.\phi_1(\alpha,\tau) - n\ P_n^{(2)}\phi_1\left(\alpha,\frac{1}{n}\right)\right\}\right)\\
&= O\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\frac{p^{(2)}_{[\tau^{-1}]}}{\delta}\int_\delta^\pi d\alpha\int_0^\delta |\chi(\alpha,\beta)|d\beta\right) + O\left(\frac{n}{P_m^{(1)}}\int_\delta^\pi d\alpha.\int^{\frac{1}{n}0}|\chi(\alpha,\beta)|d\beta\right)\\
&= O(1) + O\left(\frac{n}{P_m^{(1)}}.\frac{\frac{1}{n}}{\log\ n}\right)\\
&= O(1)\,,
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
|I_{3,2,2}| &= O\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi d\alpha\left(\frac{P^{(2)}_{[\beta^{-1}]}}{\beta}\right)\chi(\alpha,\beta)\right)\\
&= O\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\frac{1}{n}}^\delta d\left(\frac{P^{(2)}_{[\beta^{-1}]}}{\beta}\right)\int_\delta^\pi d\alpha\int_0^\beta|\chi_1(\alpha,w)|dw\right)\\
&= O\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\frac{1}{n}}^\delta d\left(\frac{P^{(2)}_{[\beta^{-1}]}}{\beta}\right)\frac{\beta}{\log(\frac{1}{\beta})}\right).
\end{aligned}
\tag{40}
$$

Also,

$$
\int_{\frac{1}{n}}^\delta \frac{\beta}{\log(\frac{1}{\beta})}d\left(P^{(2)}_{[\beta^{-1}]}\right) = \int_{\frac{1}{\Gamma}}^n \frac{1}{y.\log y}d\left(y\ P^{(2)}_{[y]}\right)\,, \tag{41}
$$

for

$$
\begin{aligned}
\int_j^{j+1}\frac{1}{y.\log y}d\left[y.P^{(2)}_{[y]}\right] &< \frac{1}{j.\log j}\int_j^{j+1}d\left[y.P^{(2)}_{[y]}\right] = \frac{1}{j.\log j}\left[y.P^{(2)}_{[y]}\right]_j^{j+1}\\
&= \frac{1}{j.\log j}\left\{(j+1)P^{(2)}_{j+1} - kP^{(2)}_j\right\}\\
&< \frac{1}{j.\log j}\left\{P^{(2)}_j + P^{(2)}_j + P^{(2)}_j\right\}.
\end{aligned}
\tag{42}
$$

For

$$
p^{(2)}_{k+1} \le p^{(2)}_k \ \text{ and } \ kp^{(2)}_k \le p^2_k \le \frac{2p^{(2)}_j}{j\log j} + \frac{p^{(2)}_j}{j.\log j}\,, \tag{43}
$$

thus,

$$
\int_{\frac{1}{\tau}}^n \frac{1}{y.\log y}d\left[xP^{(2)}_{[x]}\right] < A + \sum_{j=c}^n\left(\frac{2.P^{(2)}_k}{j\log j} + \frac{P^{(2)}_j}{j.\log j}\right) = O\left(P^{(2)}_n\right)\,. \tag{44}
$$

Now, by hypothesis (20) and using the Eqs (44) and (30), we get

$$
|I_{3,2,1}| = O(1). \tag{45}
$$

Combining (35), (36), (37), (39), (40), (44) and (45), we get

$$
|I_3| = O(1). \tag{46}
$$

Similarly, we can show that

$$|I_2| = O(1). \tag{47}$$

Now, for I_1,

$$\begin{aligned} I_1 &= \int_0^{\delta}\int_0^{\tau}\chi(\alpha,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta \\ &= \left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} + \int_0^{\frac{1}{n}}\int_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{m}}^{\delta}\int_0^{\frac{1}{n}} + \int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau}\right)\chi(\alpha\,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta).d\alpha d\beta \\ &= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}, \quad \text{(say)}. \end{aligned} \tag{48}$$

Then by (24) and (25),

$$\begin{aligned} |I_{1,1}| &= O\left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}|\chi(\alpha,\beta)|mn\, d\alpha d\beta\right) \\ &= O(mn).O\left(\frac{\frac{1}{m}}{\log m}.\frac{\frac{1}{n}}{\log n}\right) \\ &= O(1). \end{aligned} \tag{49}$$

Similarly,

$$\begin{cases} |I_{1,2}| = O(1), \\ |I_{1,3}| = O(1). \end{cases} \tag{50}$$

and

$$\begin{aligned} &\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau}\left|\chi(\alpha,\beta)\right|\frac{P_{[\alpha-1]}}{\alpha}.\frac{P_{[\beta-1]}}{\beta}d\alpha.d\beta \\ &= \chi(\delta,\tau)\frac{1}{\delta}P^{(1)}_{[\delta-1]}.\frac{1}{\tau}\,P^{(2)}_{[\tau-1]} - \frac{1}{\tau}\,P^{(2)}_{[\tau-1]} - \frac{1}{\tau}\,P^{(2)}_{[\tau-1]}.\int_{\frac{1}{m}}^{\delta}\phi(\alpha,\tau).d\left(\frac{P^{(1)}_{[\alpha-1]}}{\alpha}\right) - \frac{1}{\delta}P^{(1)}_{[\delta-1]}.\int_{\frac{1}{n}}^{\tau}\chi(\alpha,\beta)d\left(\frac{P^{(2)}_{[\beta-1]}}{\beta}\right). \end{aligned} \tag{51}$$

Thus,

$$\begin{aligned} |I_{1,4}| &= O\left(\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau}|\chi(\alpha,\beta)|.\frac{1}{P_m^{(1)}P_n^{(2)}}.\frac{P^{(2)}_{[\alpha-1]}}{\alpha}.\frac{P^{(2)}_{[\beta-1]}}{\beta}d\alpha.d\beta\right) \\ &= O(1) + O\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\left(C_1 + C_2 + C_3\right)\right), \end{aligned} \tag{52}$$

where $O(1)$ corresponds to the integrated part in (50) and C_1, C_2 and C_3 are repetitively denote the remaining there integrals. Thus following the estimates in $I_{3,2,}$, we have

$$\begin{cases} C_2 = O(1), \\ C_3 = O(1). \end{cases} \tag{53}$$

Again for C_4

$$\begin{aligned} C_4 &= O\left(\int_{\frac{1}{m}}^{\delta}\frac{\alpha}{\log(\frac{1}{\alpha})}d.\left(\frac{P^{(1)}_{[\alpha-1]}}{\alpha}\right).\int_{\frac{1}{n}}^{\tau}\frac{\beta}{\log(\frac{1}{\beta})}.\left[\frac{P^{(2)}_{[\beta-1]}}{\beta}\right]\right) \\ &= O\left(P_m^{(1)}P_n^{(2)}\right). \end{aligned} \tag{54}$$

As in (44), using the estimates (53) and (54), we get from (52) that

$$|I_{1,4}| = O(1)\,. \tag{55}$$

Thus,

$$|I_1| = O(1). \tag{56}$$

Combining (34), (45), (46), (56), we get (33). Which competes the proof of the theorem. □

Proof of Theorem 4. Let $U_{b,a}$ denote the rectangular $(b,a)^{th}$ partial sums of the series (1) Then,

$$\begin{aligned} U_{b,a} &= \int_0^{\pi}\int_0^{\pi}\chi(\alpha,\beta)N_b^{(1)}(\alpha)N_a^{(2)}(\beta)d\alpha\, d\beta \\ &= \int_0^{\frac{1}{b}}\int_0^{\frac{1}{a}} + \int_{\frac{1}{b}}^{\delta}\int_{\frac{1}{a}}^{\delta} + \int_{\delta}^{\pi}\int_0^{\pi}\chi(\alpha,\beta)N_b^{(1)}(\alpha)N_a^{(2)}(\beta)d\alpha\, d\beta \\ &= J_1 + J_2 + J_3, \quad (\text{say}). \end{aligned} \tag{57}$$

where $0 < \delta < \pi$ and $0 < \theta < \pi$. Now,

$$\left|N_b^{(1)}(\alpha)\right| \le \frac{1}{2\pi P_b^{(1)}}\sum_{l=0}^{b} p_{b-1}^{(1)}.\frac{\sin(b-l+\frac{1}{2})}{\sin\frac{\alpha}{2}}, \tag{58}$$

and

$$\left|N_b^{(2)}(\beta)\right| \le \frac{1}{2\pi P_a^{(2)}}\sum_{g=0}^{a} p_{g-a}^{(2)}.\frac{\sin(a-g+\frac{1}{2})}{\sin\frac{\beta}{2}} = O(a). \tag{59}$$

Thus

$$\begin{aligned} J_1 &= O\Big[b\,a\int_0^{\frac{1}{b}}\int_0^{\frac{1}{a}}\Big|\chi(\alpha,\beta)\Big|d\alpha d\beta\Big] \\ &= O\left(\frac{b\,p_b^{(1)}}{p_b^{(1)}}.(\frac{a\,P_a^{(2)}}{P_a^{(2)}}\right) \\ &= O\left(\frac{P_b^{(1)}}{p_b^{(1)}}.\frac{P_a^{(2)}}{p_a^{(2)}}\right) \\ &= O(1), \quad \Big(\text{because } b\,p_b^{(1)} \le P_b^{(1)} \text{and } a\,p_a^{(2)} \le P_a^{(2)}\Big)\,. \end{aligned} \tag{60}$$

Also, by Lemma 1, we have

$$\begin{aligned} J_2 &= \left[\frac{1}{p_b^{(1)}p_a^{(2)}}\int_{\frac{1}{b}}^{\delta}\int_{\frac{1}{a}}^{\theta}\chi(\alpha,\beta)\times\frac{p^{(1)}_{(\frac{1}{\alpha})}}{\alpha}.\frac{p^{(2)}_{(\frac{1}{\beta})}}{\beta}\right] \\ &= O\left[\frac{1}{p_b^{(1)}\,p_a^{(2)}}\left\{\chi(\alpha,\beta)\frac{p^{(1)}_{(\frac{1}{\alpha})}}{\alpha}\frac{p^{(2)}_{(\frac{1}{\beta})}}{\beta}\right\}_{\frac{1}{b}\frac{1}{a}}^{\delta\theta} + \frac{1}{p_b^{(1)}\,p_a^{(2)}}\int_{\frac{1}{b}}^{\delta}\int_{\frac{1}{a}}^{\theta}\chi(\alpha,\beta)\times\frac{P^{(1)}_{(\frac{1}{\alpha})}}{\alpha^2}.\frac{P^{(1)}_{(\frac{1}{\beta})}}{\beta^2}d\alpha d\beta + O(1)\right] \\ &= O(1) + \frac{1}{P_b^{(1)}p_a^{(2)}}\left\{O\left(\frac{p^{(1)}_{(\frac{1}{\alpha})}}{p^{(1)}_{(\frac{1}{\alpha})}}.\frac{p^{(2)}_{(\frac{1}{\beta})}}{p^{(2)}_{(\frac{1}{\beta})}}\right)\frac{p^{(1)}_{(\frac{1}{\alpha})}}{\alpha}.\frac{p^{(2)}_{(\frac{1}{\beta})}}{\beta}\right\}_{\frac{1}{b}\frac{1}{a}}^{\delta,\theta} + O\left[\frac{1}{p_b^{(1)}p_a^{(2)}}\int_{\frac{1}{\delta}}^{b}\int_{\frac{1}{\theta}}^{a}p^{(1)}_{(u)}p^{(2)}_{(v)}du\, dv\right] \\ &= O(1). \end{aligned} \tag{61}$$

Similarly,

$$J_3 = \int_{\delta}^{\pi} \int_{\theta}^{\pi} \chi(\alpha, \beta) N_b^{(1)}(\alpha) N_a^{(2)}(\beta) \times d\alpha\, d\beta = O(1). \tag{62}$$

Thus, combining (58), (59), (60), (61), (62), we obtain (57). This completes the proof of the theorem. □

4. Conclusion

The negligible set of conditions has been obtained for the finite series in this paper. We also discuss how Nörlund summability includes a broader range of functions that can be approximated. The approximation is a broad field having a widespread application in signals. Approximation treats signal one variable system and image as to the variable system.

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