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Closed Relations and Lyapunov Functions for Dynamical Polysystems

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

This work follows the ideas of E. Akin in an attempt to ease the construction of strict Lyapunov functions for dynamical polysystems by means of closed relations. A "best hope" type of result is presented.

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1 Introduction

The notion of *dynamical polysystem* appeared in the 1970's, being introduced by C. Lobry, [1]. It had the following meaning: a dynamical polysystem on a manifold M is a family

$$\mathcal{F}_{pc} = \{\mathcal{F}(\cdot, u) : u \in \mathcal{U}_{pc}\}$$

of smooth vector fields depending on a piecewise constant parameter u, called *input*. A similar meaning was given to dynamical polysystems in the work of J. Tsinias and N. Kalouptsidis, [2].

In this paper, a dynamical polysystem is regarded in a slightly more general way, as a family of continuous dynamical systems, all defined on the same metric space X, not necessarily by means of differential equations, more like defined by Lovingood ([3]). The analogy between dynamical polysystems and control systems with piecewise constant inputs is quite natural (also see [4]). Intuitively, a motion in a dynamical polysystem means starting at a point $x \in X$, traveling for a time t_1 according to a dynamical system Φ_1, then switching to another dynamical system Φ_2 and traveling for a time t_2, and so forth. This work is following some ideas of ([5], 1993).

2 Definitions

Consider a family $\mathcal{F}$ of continuous dynamical systems, all defined on a metric space X. For any $\phi \in \mathcal{F}$ and $t \in \mathbb{R}$, $\phi_t(x) = \phi(t, x)$ defines a homeomorphism ϕ_t on X, having inverse ϕ_{-t}.

Definition 1. *Let $\mathcal{G}$ be the subgroup of $(\mathbb{R} \times Homeo(X), (+, \circ))$ generated by $\{(t, \phi_t) : \phi \in \mathcal{F}, t \in \mathbb{R}\}$. The pair $(\mathcal{G}, X)$ is called a* ***dynamical polysystem*** *on X. The* ***accessibility semigroup*** *of $\mathcal{G}$, denoted by $\mathcal{S}$, is the subsemigroup of $\mathcal{G}$ generated by $\{(t, \phi_t) : \phi \in \mathcal{F}, t \geq 0\}$. The pair $(\mathcal{S}, X)$ is called the* ***accessibility polysystem*** *on X generated by $\mathcal{F}$.*

A similar approach (for minimal dynamical systems, see [6]) appears in [7].

Remark 1. *An element of $\mathcal{G}$ has form*

$$g = (t, h) = (t_1 + t_2 + ... + t_k, \phi^1_{t_1} \circ \phi^2_{t_2} \circ ... \circ \phi^k_{t_k}), \tag{1}$$

with $t_i \in \mathbb{R}$ and $\phi^i \in \mathcal{F}$, for $0 \leq i \leq k$.

The polysystem $(\mathcal{G}, X)$ can be considered (and, in fact, is) a $\mathcal{G}$-dynamical system. In what follows, though, notions related to dynamical systems in general may be defined or approached differently, given the concern for regarding polysystems in close connection with continuous-time dynamical systems.

3 Preliminaries

This work explores stability in dynamical polysystems by means of Lyapunov functions in a topological context, not making use of differential equations. Some other topological approaches (without explicitely using Lyapunov functions) can be found in [8], [9], and [10].

This section follows the ideas of E. Akin in an attempt to ease the problem of finding strict Lyapunov functions for polysystems. Very similar results appear, in a slightly different context, in [11]. In order to use these ideas, let us observe that a polysystem can be viewed as a closed relation, in the following sense. Define a closed relation on X by

$$f = \overline{\{(x, gx) \in X \times X : g \in \mathcal{S}_{[0,1]}\}}, \tag{2}$$

where $\mathcal{S}_{[0,1]}$ denotes all elements of $\mathcal{S}$ with time component between 0 and 1. Note that if $y = gx$, with $g \in \mathcal{S}$, then $(x, y) \in f^k$, for some positive integer k.

The facts about closed relations listed below can be found in [5].

Definition 2. *Let X be a metric space and f a closed relation on X.*

*A **Lyapunov function** for f is a continuous real-valued function L on X with the property that $L(x) \leq L(y)$ whenever $(x, y) \in f$.*

*A point $x \in X$ is **regular** for L if*

$$L(y_1) < L(x) < L(y_2) \text{ whenever } (y_1, x) \in f \text{ and } (x, y_2) \in f$$

*and **critical** for L if it is not regular.*

Denote by $|L|$ the set of critical points for L.

*Also, $|f|$ denotes the **cyclic set** of f, that is*

$$|f| := \{x \in X : (x, x) \in f\}$$

Definition 3. *Given a metric space X, a closed relation f on X, $x, y \in X$ and $\epsilon > 0$, an $\epsilon-$**chain** from x to y is a sequence of points in X, $x = x_0, x_1, ..., x_n = y$ with the property that*

$$d(x_{i+1}, f(x_i)) < \epsilon, \text{ for all } i \in \{0, ..., n-1\}.$$

Note that in the above definition $d(x_{i+1}, f(x_i))$ refers to the distance from a point to a set, which means, as usually, the infimum of distances from x_{i+1} to every point in $f(x_i)$.

Definition 4. *Given a closed relation f on a metric space X, define the **chain relation** $\mathcal{C}f$ associated to f, by $(x, y) \in \mathcal{C}f$ if for every $\epsilon > 0$, there exists an ϵ-chain from x to y.*

Note that $\mathcal{C}f$ is a closed transitive relation containing f.

Theorem 1. *(Akin, [5, pp. 33]) If F is a closed transitive relation on a compact metric space X then there exists a Lyapunov function L for F with $|L| = |F|$.*

Corollary 1. *(Akin, [5, pp. 34]) If f is a closed relation on a compact metric space X then there exists a Lyapunov function L for f with $|L| = |\mathcal{C}f|$.*

4 Polysystems Viewed as Closed Relations

Definition 5. *Let X be a metric space and $(\mathcal{S}, X)$ a polysystem, as defined in section 1. A **Lyapunov function** for the polysystem $(\mathcal{S}, X)$ is a continuous real-valued function L on X with $L(x) \leq L(gx)$ for every $x \in X$ and $g \in \mathcal{S}$.*

Remark 2. *If f is defined by 2 and L is a Lyapunov function for f then L is a Lyapunov function for the polysystem $(\mathcal{S}, X)$.*

Proof. Let L be a Lyapunov function for f, let $g \in \mathcal{S}$ and $x \in X$. Writing g as $g = g_1 g_2 ... g_k$, with $g_i \in \mathcal{S}_{[0,1]}$ for all $i \in \{1, 2, ..., k\}$, we have

$$L(gx) = L(g_1 g_2 ... g_k.x) \geq L(g_2 ... g_k.x) \geq ... \geq L(g_k.x) \geq L(x).$$

□

Definition 6. *Given $\epsilon > 0$ and $x, y \in X$, an **ϵ-chain** from x to y in the polysystem $(\mathcal{S}, X)$ is a sequence of pairs $(g_0, x_0), (g_1, x_1), ..., (g_k, x_k)$ in $(\mathcal{S}, X)$ with $x_0 = x, x_k = y$, $g_i \in \mathcal{S}_{[1,\infty)}$ for all i and $d(x_{i+1}, g_i.x_i) < \epsilon$ for all $i \in \{0, 1, ..., k\}$.*

Note that the requirement $g_i \in \mathcal{S}_{[1,\infty)}$ is needed to avoid triviality in constructing $\epsilon-$chains. Without it, any two points in X could be connected through an $\epsilon-$chain, using the mere continuity of actions by elements in $\mathcal{S}$ on X.

Finally, define a chain relation $\mathcal{C}$ for the polysystem $(\mathcal{S}, X)$, by

$$(x, y) \in \mathcal{C} \text{ if for every } \epsilon > 0 \text{ there exists an } \epsilon - chain \text{ from } x \text{ to } y, \tag{3}$$

(in the sense of polysystems).

Definition 7. *A point x in X is said to be **chain-recurrent** (in the sense of polysystems) if $x \in |\mathcal{C}|$, (that is, for every $\epsilon > 0$ there exists an ϵ-chain from x to x).*

Proposition 1. *If f is defined by 2 and $\mathcal{C}$ by 3 then $\mathcal{C} \subset \mathcal{C}f$.*

Proof. Let $(x, y) \in \mathcal{C}$. For $\epsilon > 0$ there exists an $\epsilon - chain$ (in the sense of polysystems) from x to y, $(g_0, x_0), (g_1, x_1), ..., (g_k, x_k)$. Every g_i in this chain can be written as

$$g_i = g_i^{j_1} g_i^{j_2} ... g_i^{j_{k_i}}$$

with $g_i^{j_l} \in \mathcal{S}_{[0,1]}$, for all l. We can construct then an $\epsilon - chain$ from x to y (in the sense of relations), as follows:

$$x = x_0, ..., g_{i-1}x_{i-1}, x_i, g_i^{j_{k_i}} x_i, g_i^{j_{k_i-1}} g_i^{j_{k_i}} x_i, ..., g_i^{j_1} g_i^{j_2} ... g_i^{j_{k_i}} x_i = g_i x_i, x_{i+1}, ..., ..., x_k.$$

It suffices to show now that $d(g_{i-1}x_{i-1}, f(x_i)) < \epsilon$ and $d(g_i^{j_{k_i}} x_i, f(x_i)) < \epsilon$. The first inequality is seen to be satisfied by noting that $d(g_{i-1}x_{i-1}, x_i) < \epsilon$ and $x_i \in f(x_i)$. The second one is true since $g_i^{j_{k_i}} x_i \in f(x_i)$ and so $d(g_i^{j_{k_i}} x_i, f(x_i)) = 0 < \epsilon$.

□

Theorem 2. *If $(\mathcal{S}, X)$ is a polysystem defined on the compact metric space X then there exists a Lyapunov function L for the polysystem with $|L| = |\mathcal{C}f|$.*

Proof. The theorem follows from Corollary 1. □

Corollary 2. *If $(\mathcal{S}, X)$ is a polysystem defined on the compact metric space X then there exists a Lyapunov function L for the polysystem with $|\mathcal{C}| \subset |L|$.*

5 Conclusion

From this Corollary we draw the conclusion that, in trying to obtain a strict Lyapunov function L for the polysystem $(\mathcal{S}, X)$, the most one can hope is that the critical points for L are precisely the chain-recurrent points in the polysystem.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Lobry C. Dynamical polysystems and control theory. D. Reidel Publishing Company, Dordrecht, NL; 1973.

[2] Tsinias J, Kalouptsidis N. Prolongations and Stability Analysis via Lyapunov Functions of Dynamical Polysystems. Math. Systems Theory 20. 1987;2-3:215–233.

[3] Lovingood JA. A Special Class of Dynamical Polysystems. Journal of Differential Equations. 1969;6:326-336.

[4] Bacciotti A, Mazzi L. Stability of dynamical polysystems via families of Liapunov functions. Nonlinear Analysis: Theory, Methods & Applications. 2007;67(7):2167-2179.

[5] Akin E. The General Topology of Dynamical Systems, Volume I, American Mathematical Society; 1993.

[6] Kolyada S, Snoha L. Minimal dynamical systems, Scholarpedia. 2009;4(11):5803.

[7] Iztok Banic, Goran Erceg, Rene Gril Rogina, Judy Kennedy. Minimal dynamical systems with closed relations. arXiv:2205.02907v1 [math.DS]; 2022.

[8] Victor HL. Rocha. Lyapunov stability and weak attraction for control systems. Proyecciones Journal of Mathematics. 2022;41(3):605-622.

[9] Braga Barros CJ, Souza JA, Rocha VHL. Lyapunov stability and attraction under equivariant maps. Canadian Journal of Mathematics. 2015;67(6):1247-1269.

[10] Souza JA. Complete Lyapunov functions of control systems. Systems & Control Letters. 2012;61(2):322-326.

[11] Gui Seok Kim, Kyung Bok Lee, Jong Suh Park. Attractors and Lyapunov function for closed relations. Commun. Korean Math. Soc. 2014; 29(1): 163-172.

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