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Computing the Determinant and Inverse of the Complex Fibonacci Hermitian Toeplitz Matrix

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Authors' contributions

This work was carried out in collaboration by both authors. Author ZJ designed the study, proposed the concerned problem. Author JS managed the analyses of the study and wrote the first draft of the manuscript. Both authors read and approved the final manuscript.

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Abstract

In this paper, we consider the determinant and the inverse of the complex Fibonacci Hermitian Toeplitz matrix. We first give the definition of the complex Fibonacci Hermitian Toeplitz matrix. Then we compute the determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix by constructing the transformation matrices.

Keywords: Hermitian Toeplitz matrix; determinant; inverse; complex Fibonacci numbers.

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1 Introduction

Studies show that there has been an increasing interest on Fibonacci sequence and its generalizations. One of them is the concept of complex Fibonacci numbers F_n^*, which was defined by Horadam [1]. The n-th complex Fibonacci number is given by the equality $F_n^* = F_n + iF_{n+1}$, where i is the imaginary unit which satisfies $i^2 = -1$. Complex Fibonacci numbers satisfy the same recurrence relation $F_n^* = F_{n-1}^* + F_{n-2}^* (n \geqslant 2)$ of the classical Fibonacci numbers with different initial conditions, i.e., $F_0^* = i, F_1^* = 1 + i$.

On the other hand, Hermitian Toeplitz matrices have important applications in various disciplines including image processing, signal processing, and solving least squares problems [2, 3, 4].

It is an ideal research area and hot topic for the inverses of the special matrices with famous numbers. Some scholars showed the explicit determinants and inverses of the special matrices involving famous numbers. Lin showed the determinant of the Fibonacci-Lucas quasi-cyclic matrices in [5]. Circulant matrices with Fibonacci and Lucas numbers are discussed and their explicit determinants and inverses are proposed in [6]. The authors provided determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [7]. In [8], circulant type matrices with the k-Fibonacci and k-Lucas numbers are considered and the explicit determinants and inverse matrices are presented by constructing the transformation matrices. The explicit determinants of circulant and left circulant matrices including Tribonacci numbers and generalized Lucas numbers are shown based on Tribonacci numbers and generalized Lucas numbers in [9]. In [10], Jiang and Hong gave the exact determinants of the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Padovan, Perrin, Tribonacci and the generalized Lucas numbers by the inverse factorization of polynomial. Jiang et al. [11] gave the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and provided the determinants and the inverses of these matrices. It should be noted that Jiang and Zhou [12] obtained the explicit formula for spectral norm of an r-circulant matrix whose entries in the first row are alternately positive and negative, and the authors [13] investigated explicit formulas of spectral norms for g-circulant matrices with Fibonacci and Lucas numbers. Furthermore, in [14] the determinants and inverses are discussed and evaluated for Tribonacci skew circulant type matrices. The authors [15] proposed the invertibility criterium of the generalized Lucas skew circulant type matrices and provided their determinants and the inverse matrices. The determinants and inverses of Tribonacci circulant type matrices are discussed in [16]. Determinants and inverses of Fibonacci and Lucas skew symmetric Toeplitz matrices are given by constructing the special transformation matrices in [17].

The purpose of this paper is to obtain better results for the determinants and inverses of complex Fibonacci Hermitian Toeplitz matrix. In this paper we adopt the following two conventions $0^0 = 1$, $i^2 = -1$, and we define a kind of special matrix as follows.

Definition 1.1. A complex Fibonacci Hermitian Toeplitz matrix is a square matrix of the form

$$\mathbf{T}_{F_n^*} = \begin{pmatrix} 0 & F_0^* & F_1^* & \cdots & F_{n-4}^* & F_{n-3}^* & F_{n-2}^* \\ \bar{F}_0^* & 0 & F_0^* & \cdots & F_{n-5}^* & F_{n-4}^* & F_{n-3}^* \\ \bar{F}_1^* & \bar{F}_0^* & 0 & \cdots & F_{n-6}^* & F_{n-5}^* & F_{n-4}^* \\ & \cdots & & \cdots & & \cdots & \\ \bar{F}_{n-4}^* & \bar{F}_{n-5}^* & \bar{F}_{n-6}^* & \cdots & 0 & F_0^* & F_1^* \\ \bar{F}_{n-3}^* & \bar{F}_{n-4}^* & \bar{F}_{n-5}^* & \cdots & \bar{F}_0^* & 0 & F_0^* \\ \bar{F}_{n-2}^* & \bar{F}_{n-3}^* & \bar{F}_{n-4}^* & \cdots & \bar{F}_1^* & \bar{F}_0^* & 0 \end{pmatrix}_{n\times n}, \tag{1.1}$$

where $F_0^*,\ F_1^*, \cdots,\ F_{n-2}^*$ are the complex Fibonacci numbers.

It is evidently determined by its first row and first column, and $\mathbf{T}_{F_n^*} = \bar{\mathbf{T}}_{F_n^*}^T$.

Lemma 1.1. *Let p_i be a Hessenberg matrix,*

$$p_i = \begin{pmatrix} \kappa_2 & \kappa_1 & 0 & \cdots & \cdots & \cdots & 0 \\ \kappa_3 & \kappa_2 & \kappa_1 & \ddots & & & \vdots \\ \kappa_4 & \kappa_3 & \kappa_2 & \kappa_1 & \ddots & & \vdots \\ \kappa_5 & \kappa_4 & \kappa_3 & \kappa_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \kappa_1 \\ \kappa_{i+1} & \cdots & \cdots & \kappa_5 & \kappa_4 & \kappa_3 & \kappa_2 \end{pmatrix}_{i\times i},$$

then the determinant of p_i can be expressed as

$$\det p_i = \sum_{j=1}^{i} (-1)^{i+j} \kappa_{i-j+2} \kappa_1^{i-j} \det p_{j-1}, \ i \geqslant 2,$$

where the initial values of $\det p_i$ are $\det p_0 = 1$ and $\det p_1 = \kappa_2$.

Proof. By using the Laplace expansion of matrix p_i along the last row, it is easy to check that

$$\begin{aligned} \det p_i &= (-1)^{i+1} \kappa_{i+1} \kappa_1^{i-1} \det p_0 + (-1)^{i+2} \kappa_i \kappa_1^{i-2} \det p_1 \\ &+ (-1)^{i+3} \kappa_{i-1} \kappa_1^{i-3} \det p_2 + \cdots + (-1)^{i+i-1} \kappa_3 \kappa_1 \det p_{i-2} \\ &+ (-1)^{i+i} \kappa_2 \kappa_1^{0} \det p_{i-1} \\ &= \sum_{j=1}^{i} (-1)^{i+j} \kappa_{i-j+2} \kappa_1^{i-j} \det p_{j-1}, \end{aligned}$$

for $i \geqslant 2$. The initial values of $\det p_i$ are $\det p_0 = 1$ and $\det p_1 = \kappa_2$. □

Lemma 1.2. *Define an $i \times i$ Toeplitz-like matrix by*

$$\nabla_i([\mu_k]_{k=1}^{i}, \alpha_1, \alpha_2, \cdots, \alpha_i) = \begin{pmatrix} \mu_i & \mu_{i-1} & \mu_{i-2} & \mu_{i-3} & \mu_{i-4} & \cdots & \mu_2 & \mu_1 \\ \alpha_2 & \alpha_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \ddots & & & & \vdots \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \ddots & & & \vdots \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ \alpha_i & \cdots & \cdots & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}_{i\times i},$$

we have

$$\det \nabla_i([\mu_k]_{k=1}^{i}, \alpha_1, \alpha_2, \cdots, \alpha_i) = \sum_{j=1}^{i} (-1)^{1+j} \mu_{i+1-j} \alpha_1^{i-j} \det p_{j-1},$$

where $\det p_j (0 \leqslant j \leqslant i-1)$ is the determinant of Hessenberg matrix, and can be calculated by Lemma 1.1.

Proof. By using the Laplace expansion of matrix $\nabla_i([\mu_k]_{k=1}^i, \alpha_1, \alpha_2, \cdots, \alpha_i)$ along the first row, it is easy to check that

$$\begin{aligned}\det \nabla_i([\mu_k]_{k=1}^i, \alpha_1, \alpha_2, \cdots, \alpha_i) &= (-1)^{1+1}\mu_i\alpha_1^{i-1}\det p_0 + (-1)^{1+2}\mu_{i-1}\alpha_1^{i-2}\det p_1 + (-1)^{1+3}\mu_{i-2}\alpha_1^{i-3}\det p_2 \\ &\quad + \cdots + (-1)^{1+i-1}\mu_2\alpha_1 \det p_{i-2} + (-1)^{1+i}\mu_1\alpha_1^0 \det p_{i-1} \\ &= \sum_{j=1}^{i}(-1)^{1+j}\mu_{i-j+1}\alpha_1^{i-j}\det p_{j-1},\end{aligned}$$

where $\det p_j (0 \leqslant j \leqslant i-1)$ is the determinant of Hessenberg matrix, and can be calculated by Lemma 1.1. □

2 Determinant and Inverse of the Complex Fibonacci Hermitian Toeplitz Matrix

Let $\mathbf{T}_{F_n^*}$ be a complex Fibonacci Hermitian Toeplitz matrix. In this section, we first give the determinant of the matrix $\mathbf{T}_{F_n^*}$, and then we find the inverse of $\mathbf{T}_{F_n^*}$, assuming that it is invertible.

Theorem 2.1. *Let $\mathbf{T}_{F_n^*}$ be a complex Fibonacci Hermitian Toeplitz matrix as the form of (1.1). Then we have*

$$\det \mathbf{T}_{F_1^*} = 0,\ \det \mathbf{T}_{F_2^*} = -1,\ \det \mathbf{T}_{F_3^*} = -2,\ \det \mathbf{T}_{F_4^*} = 6,\ \det \mathbf{T}_{F_5^*} = -2$$

and

$$\begin{aligned}\det \mathbf{T}_{F_n^*} = -\bar{F}_0^*[&F_0^*(K_3 \det \nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}) \\ &- K_4 \det \nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})) \\ &- \bar{F}_{n-3}^*(K_1 \det \nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}) \\ &- K_4 \det \nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})) \\ &+ \bar{F}_{n-4}^*(K_1 \det \nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}) \\ &- K_3 \det \nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}))],\ n > 5,\end{aligned} \tag{2.1}$$

where

$K_1 = \sum_{k=1}^{n-2} F_k^* \Delta_{n-2-k},\ K_3 = \sum_{k=1}^{n-2} \xi_k \Delta_{n-2-k},\ K_4 = \sum_{k=1}^{n-2} \eta_k \Delta_{n-2-k},$

$\Delta_0 = 1,\ \Delta_1$ *is one root of the characteristic equation* $-x^2 + \alpha x + t_0 = 0,$

$\Delta_2 = \Delta_1^2,\ \Delta_i = \alpha\Delta_{i-1} + \sum_{k=0}^{i-2} t_k \Delta_{i-2-k},\ (3 \leqslant i \leqslant n-3)$

$\xi_i = xF_{i-1}^* + \bar{F}_{n-i-3}^*,\ (1 \leqslant i \leqslant n-3),\ \xi_{n-2} = xF_{n-3}^*,$

$\eta_i = yF_{i-1}^* + \bar{F}_{n-4-i},\ (1 \leqslant i \leqslant n-4),\ \eta_{n-3} = yF_{n-4}^*,\ \eta_{n-2} = yF_{n-3}^* + F_0^*,$

$\alpha = 2i,\ t_0 = F_2^*,\ t_i = 2F_i^*,\ (1 \leqslant i \leqslant n-5),\ x = -\frac{\bar{F}_{n-2}^*}{\bar{F}_0^*},\ y = -\frac{\bar{F}_{n-3}^*}{\bar{F}_0^*},$

$\det \nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}),\ \det \nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})$ *and* $\det \nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})$ *can be calculated by Lemma 1.2.*

Proof. Let $\mathbf{T}_{F_n^*}$ be a complex Fibonacci Hermitian Toeplitz matrix. We can easily get the following conclusions:

$$\det \mathbf{T}_{F_1^*} = 0,\ \det \mathbf{T}_{F_2^*} = -1,\ \det \mathbf{T}_{F_3^*} = -2,\ \det \mathbf{T}_{F_4^*} = 6,\ \det \mathbf{T}_{F_5^*} = -2.$$

We can introduce the following two transformation matrices when $n > 5$,

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & x & \vdots & & & & & \iddots & 1 \\ \vdots & y & \vdots & & & & \iddots & 1 & 0 \\ \vdots & 0 & \vdots & & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & \vdots & & \iddots & 1 & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \vdots & 0 & 1 & 1 & -1 & \iddots & & \vdots \\ 0 & 0 & 1 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n\times n},$$

and

$$\mathcal{N}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \Delta_{n-3} & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \Delta_{n-4} & \vdots & & & \iddots & 1 & 0 \\ \vdots & \vdots & \Delta_{n-5} & \vdots & & \iddots & 1 & \iddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \vdots & \Delta_2 & 0 & \iddots & \iddots & & & \vdots \\ \vdots & \vdots & \Delta_1 & 1 & \iddots & & & & \vdots \\ 0 & 0 & \Delta_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n},$$

where

$$x = -\frac{\bar{F}_{n-2}^*}{\bar{F}_0^*},\ y = -\frac{\bar{F}_{n-3}^*}{\bar{F}_0^*},\ \Delta_0 = 1,$$

Δ_1 is one root of the characteristic equation $-x^2 + \alpha x + t_0 = 0$,

$$\Delta_2 = \Delta_1^2,\ \Delta_i = \alpha\Delta_{i-1} + \sum_{k=0}^{i-2} t_k \Delta_{i-2-k},\ (3 \leqslant i \leqslant n-3),$$

$$\alpha = 2i,\ t_0 = F_2^*,\ t_i = 2F_i^*,\ (1 \leqslant i \leqslant n-5).$$

By using $\mathcal{M}_1$, $\mathcal{N}_1$ and the recurrence relations of the complex Fibonacci sequences, the matrix $\mathbf{T}_{F_n^*}$

is changed into the following form,

$$\mathcal{M}_1\mathbf{T}_{F_n^*}\mathcal{N}_1 = \begin{pmatrix} 0 & F_0^* & K_1 & F_{n-3}^* & F_{n-4}^* & F_{n-5}^* & F_{n-6}^* & \cdots & F_2^* & F_1^* \\ \bar{F}_0^* & 0 & K_2 & F_{n-4}^* & F_{n-5}^* & F_{n-6}^* & F_{n-7}^* & \cdots & F_1^* & F_0^* \\ 0 & \bar{F}_{n-3}^* & K_3 & \xi_{n-3} & \xi_{n-4} & \xi_{n-5} & \xi_{n-6} & \cdots & \xi_2 & \xi_1 \\ \vdots & \bar{F}_{n-4}^* & K_4 & \eta_{n-3} & \eta_{n-4} & \eta_{n-5} & \eta_{n-6} & \cdots & \eta_2 & \eta_1 \\ \vdots & 0 & 0 & \alpha & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & t_0 & \alpha & -1 & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & t_1 & t_0 & \alpha & -1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & t_{n-6} & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix}_{n\times n},$$

where

$$\xi_i = xF_{i-1}^* + \bar{F}_{n-i-3}^*,\ (1 \leqslant i \leqslant n-3),\ \xi_{n-2} = xF_{n-3}^*,$$
$$\eta_i = yF_{i-1}^* + \bar{F}_{n-4-i},\ (1 \leqslant i \leqslant n-4),\ \eta_{n-3} = yF_{n-4}^*,\ \eta_{n-2} = yF_{n-3}^* + F_0^*,$$
$$\alpha = 2i,\ t_0 = F_2^*,\ t_i = 2F_i^*,\ (1 \leqslant i \leqslant n-5),$$
$$K_1 = \sum_{k=1}^{n-2} F_k^*\Delta_{n-2-k},\ K_2 = \sum_{k=0}^{n-3} F_k^*\Delta_{n-3-k},\ K_3 = \sum_{k=1}^{n-2} \xi_k\Delta_{n-2-k},\ K_4 = \sum_{k=1}^{n-2} \eta_k\Delta_{n-2-k}.$$

By using the Laplace expansion of matrix $\mathcal{M}_1\mathbf{T}_{F_n^*}\mathcal{N}_1$ along the first column, we can get that

$$\begin{aligned}\det(\mathcal{M}_1\mathbf{T}_{F_n^*}\mathcal{N}_1) = -\bar{F}_0^*[F_0^*(&K_3 \det\nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}) \\ &- K_4 \det\nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})) \\ &- \bar{F}_{n-3}^*(K_1 \det\nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}) \\ &- K_4 \det\nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})) \\ &+ \bar{F}_{n-4}^*(K_1 \det\nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}) \\ &- K_3 \det\nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6}))],\end{aligned}$$

where

$$\nabla_{n-3}([\eta_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})$$
$$= \begin{pmatrix} \eta_{n-3} & \eta_{n-4} & \eta_{n-5} & \eta_{n-6} & \cdots & \eta_4 & \eta_3 & \eta_2 & \eta_1 \\ \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ t_0 & \alpha & -1 & \ddots & & & & & \vdots \\ t_1 & t_0 & \alpha & -1 & \ddots & & & & \vdots \\ t_2 & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \alpha & -1 & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & t_0 & \alpha & -1 & 0 \\ t_{n-6} & t_{n-7} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix}_{(n-3)\times(n-3)},$$

$$\nabla_{n-3}([\xi_k]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})$$
$$= \begin{pmatrix} \xi_{n-3} & \xi_{n-4} & \xi_{n-5} & \xi_{n-6} & \cdots & \xi_4 & \xi_3 & \xi_2 & \xi_1 \\ \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ t_0 & \alpha & -1 & \ddots & & & & & \vdots \\ t_1 & t_0 & \alpha & -1 & \ddots & & & & \vdots \\ t_2 & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \alpha & -1 & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & t_0 & \alpha & -1 & 0 \\ t_{n-6} & t_{n-7} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix}_{(n-3)\times(n-3)},$$

$$\nabla_{n-3}([F_k^*]_{k=1}^{n-3}, -1, \alpha, t_0, t_1, \cdots, t_{n-6})$$
$$= \begin{pmatrix} F_{n-3}^* & F_{n-4}^* & F_{n-5}^* & F_{n-6}^* & \cdots & F_4^* & F_3^* & F_2^* & F_1^* \\ \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ t_0 & \alpha & -1 & \ddots & & & & & \vdots \\ t_1 & t_0 & \alpha & -1 & \ddots & & & & \vdots \\ t_2 & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \alpha & -1 & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & t_0 & \alpha & -1 & 0 \\ t_{n-6} & t_{n-7} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix}_{(n-3)\times(n-3)}$$

and the determinant of them can be calculated by Lemma 1.2.

While

$$\det \mathcal{M}_1 = \det \mathcal{N}_1 = (-1)^{\frac{(n+1)(n-2)}{2}},$$

we can obtain $\det \mathbf{T}_{F_n^*}$ as (2.1), which completes the proof. □

Theorem 2.2. *Let $\mathbf{T}_{F_n^*}$ be an invertible complex Fibonacci Hermitian Toeplitz matrix and $n > 6$. Then we have*

$$\mathbf{T}_{F_n^*}^{-1} = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} & \cdots & \psi_{1,n-2} & \psi_{1,n-1} & \psi_{1,n} \\ \bar{\psi}_{1,2} & \psi_{2,2} & \psi_{2,3} & \cdots & \psi_{2,n-2} & \psi_{2,n-1} & \psi_{1,n-1} \\ \bar{\psi}_{1,3} & \bar{\psi}_{2,3} & \psi_{3,3} & \cdots & \psi_{3,n-3} & \psi_{2,n-2} & \psi_{1,n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \bar{\psi}_{1,n-2} & \bar{\psi}_{2,n-2} & \bar{\psi}_{3,n-3} & \cdots & \psi_{3,3} & \psi_{2,3} & \psi_{1,3} \\ \bar{\psi}_{1,n-1} & \bar{\psi}_{2,n-1} & \bar{\psi}_{2,n-2} & \cdots & \bar{\psi}_{2,3} & \psi_{2,2} & \psi_{1,2} \\ \bar{\psi}_{1,n} & \bar{\psi}_{1,n-1} & \bar{\psi}_{1,n-2} & \cdots & \bar{\psi}_{1,3} & \bar{\psi}_{1,2} & \psi_{1,1} \end{pmatrix}, \quad (2.2)$$

where

$$\psi_{1,1}=\frac{K_2}{\bar{F}_0^*}\left(\frac{\bar{F}_{n-3}^*}{F_0^*}a_{1,1}+\frac{\bar{F}_{n-4}^*}{F_0^*}a_{1,2}\right)+\sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}\left(\frac{\bar{F}_{n-3}^*}{F_0^*}a_{n-2-k,1}+\frac{\bar{F}_{n-4}^*}{F_0^*}a_{n-2-k,2}\right),$$

$$\psi_{1,2}=\frac{1}{\bar{F}_0^*}-\frac{K_2}{\bar{F}_0^*}\left(xa_{1,1}+ya_{1,2}\right)-\sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}\left(xa_{n-2-k,1}+ya_{n-2-k,2}\right),$$

$$\psi_{1,3}=-\frac{K_2}{\bar{F}_0^*}a_{1,n-2}-\sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}a_{n-2-k,n-2},$$

$$\psi_{1,4}=-\frac{K_2}{\bar{F}_0^*}(a_{1,n-3}+a_{1,n-2})-\sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}(a_{n-2-k,n-3}+a_{n-2-k,n-2}),$$

$$\psi_{1,j}=-\frac{K_2}{\bar{F}_0^*}(a_{1,n+1-j}+a_{1,n+2-j}-a_{1,n+3-j})$$
$$-\sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}(a_{n-2-k,n+1-j}+a_{n-2-k,n+2-j}-a_{n-2-k,n+3-j}),\ (5\leqslant j\leqslant n-1),$$

$$\psi_{1,n}=-\frac{K_2}{\bar{F}_0^*}(a_{1,1}-a_{1,3})-\sum_{k=0}^{n-4}\frac{F_k^*}{\bar{F}_0^*}(a_{n-2-k,1}-a_{n-2-k,3}),$$

$$\psi_{2,2}=-\frac{K_1}{F_0^*}\left(xa_{1,1}+ya_{1,2}\right)-\sum_{k=1}^{n-3}\frac{F_k^*}{F_0^*}\left(xa_{n-1-k,1}+ya_{n-1-k,2}\right),$$

$$\psi_{2,3}=-\frac{K_1}{F_0^*}a_{1,n-2}-\sum_{k=1}^{n-3}\frac{F_k^*}{F_0^*}a_{n-1-k,n-2},$$

$$\psi_{2,4}=-\frac{K_1}{F_0^*}(a_{1,n-3}+a_{1,n-2})-\sum_{k=1}^{n-3}\frac{F_k^*}{F_0^*}(a_{n-1-k,n-3}+a_{n-1-k,n-2}),$$

$$\psi_{2,j}=-\frac{K_1}{F_0^*}(a_{1,n+1-j}+a_{1,n+2-j}-a_{1,n+3-j})$$
$$-\sum_{k=1}^{n-3}\frac{F_k^*}{F_0^*}(a_{n-1-k,n+1-j}+a_{n-1-k,n+2-j}-a_{n-1-k,n+3-j}),\ (5\leqslant j\leqslant n-1),$$

$$\psi_{3,3}=\Delta_{n-3}a_{1,n-2}+a_{n-2,n-2},$$

$$\psi_{i,4}=\Delta_{n-i}(a_{1,n-3}+a_{1,n-2})+(a_{n-i+1,n-3}+a_{n-i+1,n-2}),\ (3\leqslant i\leqslant 4),$$

$$\psi_{i,j}=\Delta_{n-i}(a_{1,n+1-j}+a_{1,n+2-j}-a_{1,n+3-j})+(a_{n-i+1,n+1-j}+a_{n-i+1,n+2-j}-a_{n-i+1,n+3-j}),$$
$$(3\leqslant i\leqslant[\frac{n+1}{2}];\ i\leqslant j\leqslant n+1-i;\ \textit{except}\ \psi_{3,3},\ \psi_{3,4}\ \textit{and}\ \psi_{4,4}),$$

in which $a_{i,j}$ *is defined by*

$$a_{1,1}=\frac{1}{s}\left(\omega_{n-3}-\omega_{n-4}b_1\alpha-\sum_{k=1}^{n-5}\omega_k(b_{n-3-k}\alpha+\sum_{j=1}^{n-4-k}b_jt_{n-4-k-j})\right),$$

$$a_{1,2} = -\frac{1}{s}\left(\beta_{n-3} - \beta_{n-4}b_1\alpha - \sum_{k=1}^{n-5}\beta_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right), a_{2,1} = -\frac{d}{s},\ a_{2,2} = \frac{c}{s},$$

$$a_{i,j} = \begin{cases} -\sum_{k=1}^{n-j-1} b_k(a_{i,1}\beta_{n-j-k} + a_{i,2}\omega_{n-j-k}),\ (1 \leqslant i \leqslant 2,\ 3 \leqslant j \leqslant n-2), \\ -b_1\alpha a_{2,j},\ (i=3,\ 1 \leqslant j \leqslant 2), \\ -a_{2,j}(b_{i-2}\alpha + \sum_{k=1}^{i-3} b_k t_{i-3-k}),\ (4 \leqslant i \leqslant n-2,\ 1 \leqslant j \leqslant 2), \\ b_1\alpha\left(\sum_{k=1}^{n-4} b_k(a_{2,1}\beta_{n-3-k} + a_{2,2}\omega_{n-3-k})\right) + b_1,\ (i=3,\ j=3), \\ b_1\alpha\left(\sum_{k=1}^{n-1-j} b_k(a_{2,1}\beta_{n-j-k} + a_{2,2}\omega_{n-j-k})\right),\ (i=3,\ 4 \leqslant j \leqslant n-2), \\ (b_{i-2}\alpha + \sum_{k=1}^{i-3} b_k t_{i-3-k})\left(\sum_{k=1}^{n-1-j} b_k(a_{2,1}\beta_{n-j-k} + a_{2,2}\omega_{n-j-k})\right) + b_{i-j+1}, \\ \qquad (4 \leqslant i \leqslant n-2,\ 3 \leqslant j \leqslant n-2,\ i \geqslant j), \\ (b_{i-2}\alpha + \sum_{k=1}^{i-3} b_k t_{i-3-k})\left(\sum_{k=1}^{n-1-j} b_k(a_{2,1}\beta_{n-j-k} + a_{2,2}\omega_{n-j-k})\right), \\ \qquad (4 \leqslant i \leqslant n-2,\ 3 \leqslant j \leqslant n-2,\ i < j). \end{cases}$$

with

$$\begin{aligned} s &= c\left(\omega_{n-3} - \omega_{n-4}b_1\alpha - \sum_{k=1}^{n-5}\omega_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right) \\ &\quad - d\left(\beta_{n-3} - \beta_{n-4}b_1\alpha - \sum_{k=1}^{n-5}\beta_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right), \\ b_i &= -\det p_{i-1},\ (1 \leqslant i \leqslant n-4),\ c = -\frac{\bar{F}^*_{n-3}}{F^*_0}K_1 + K_3,\ d = -\frac{\bar{F}^*_{n-4}}{F^*_0}K_1 + K_4, \\ \beta_i &= -\frac{\bar{F}^*_{n-3}}{F^*_0}F^*_i + \xi_i,\ \omega_i = -\frac{\bar{F}^*_{n-4}}{F^*_0}F^*_i + \eta_i,\ (1 \leqslant i \leqslant n-3), \end{aligned}$$

K_1, K_2, K_3, K_4, x, y, α, $\Delta_i(0 \leqslant i \leqslant n-3)$, $\xi_i(1 \leqslant i \leqslant n-3)$, $\eta_i(1 \leqslant i \leqslant n-3)$ *and* $t_i(0 \leqslant i \leqslant n-6)$ *are as in Theorem 2.1,* $\det p_i(0 \leqslant i \leqslant n-5)$ *is the determinant of Hessenberg matrix, and can be calculated by Lemma 1.1,* $[x] = m,\ m \leqslant x < m+1,\ m$ *is an integer.*

We can observe that $\mathbf{T}^{-1}_{F^*_n}$ *is not only a Hermitian matrix, but also a symmetric matrix along its secondary diagonal, i.e., sub-symmetric matrix.*

Proof. We can introduce the following two transformation matrices when $n > 6$,

$$\mathcal{M}_2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & & \vdots \\ -\frac{\bar{F}^*_{n-3}}{F^*_0} & 0 & 1 & \ddots & & & & & \vdots \\ -\frac{\bar{F}^*_{n-4}}{F^*_0} & \vdots & \ddots & 1 & \ddots & & & & \vdots \\ 0 & \vdots & & \ddots & 1 & \ddots & & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & & \ddots & 1 & \ddots & \vdots \\ \vdots & \vdots & & & & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

$$\mathcal{N}_2 = \begin{pmatrix} 1 & 0 & -\frac{K_2}{\bar{F}^*_0} & -\frac{F^*_{n-4}}{\bar{F}^*_0} & -\frac{F^*_{n-5}}{\bar{F}^*_0} & \cdots & -\frac{F^*_1}{\bar{F}^*_0} & -\frac{F^*_0}{\bar{F}^*_0} \\ 0 & 1 & -\frac{K_1}{F^*_0} & -\frac{F^*_{n-3}}{F^*_0} & -\frac{F^*_{n-4}}{F^*_0} & \cdots & -\frac{F^*_2}{F^*_0} & -\frac{F^*_1}{F^*_0} \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & \ddots & 1 & \ddots & & & \vdots \\ \vdots & & & \ddots & 1 & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

If we multiply $\mathcal{M}_1\mathbf{T}_{F^*_n}\mathcal{N}_1$ by $\mathcal{M}_2$ and $\mathcal{N}_2$, the $\mathcal{M}_1$ and $\mathcal{N}_1$ are as in the proof of Theorem 2.1, so we obtain

$$\mathcal{M}_2\mathcal{M}_1\mathbf{T}_{F^*_n}\mathcal{N}_1\mathcal{N}_2 = \begin{pmatrix} 0 & F^*_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \bar{F}^*_0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & c & \beta_{n-3} & \beta_{n-4} & \beta_{n-5} & \cdots & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ \vdots & \vdots & d & \omega_{n-3} & \omega_{n-4} & \omega_{n-5} & \cdots & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\ \vdots & \vdots & 0 & \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & t_0 & \alpha & -1 & \ddots & & & & \vdots \\ \vdots & \vdots & \vdots & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & t_{n-6} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix},$$

with

$$c = -\frac{\bar{F}^*_{n-3}}{F^*_0}K_1 + K_3,\ \beta_i = -\frac{\bar{F}^*_{n-3}}{F^*_0}F^*_i + \xi_i,\ (1 \leqslant i \leqslant n-3),$$
$$d = -\frac{\bar{F}^*_{n-4}}{F^*_0}K_1 + K_4,\ \omega_i = -\frac{\bar{F}^*_{n-4}}{F^*_0}F^*_i + \eta_i,\ (1 \leqslant i \leqslant n-3).$$

We have

$$\mathcal{M}_2\mathcal{M}_1\mathbf{T}_{F^*_n}\mathcal{N}_1\mathcal{N}_2 = \mathbf{\Phi} \oplus \mathbf{\Lambda},$$

where $\mathbf{\Phi} = \begin{pmatrix} 0 & F^*_0 \\ \bar{F}^*_0 & 0 \end{pmatrix}_{2\times 2}$ and $\mathbf{\Lambda}$ is a Toeplitz-like matrix

$$\mathbf{\Lambda} = \begin{pmatrix} c & \beta_{n-3} & \beta_{n-4} & \beta_{n-5} & \cdots & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ d & \omega_{n-3} & \omega_{n-4} & \omega_{n-5} & \cdots & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\ 0 & \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & t_0 & \alpha & -1 & \ddots & & & & 0 \\ \vdots & t_1 & t_0 & \alpha & \ddots & \ddots & & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & t_{n-6} & \cdots & \cdots & \cdots & t_1 & t_0 & \alpha & -1 \end{pmatrix}_{(n-2)\times(n-2)}.$$

$\mathbf{\Phi} \oplus \mathbf{\Lambda}$ is the direct sum of $\mathbf{\Phi}$ and $\mathbf{\Lambda}$. Let $\mathcal{M} = \mathcal{M}_2\mathcal{M}_1$, $\mathcal{N} = \mathcal{N}_1\mathcal{N}_2$, then we obtain

$$\mathbf{T}^{-1}_{F^*_n} = \mathcal{N}(\mathbf{\Phi}^{-1} \oplus \mathbf{\Lambda}^{-1})\mathcal{M},$$

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \vdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\frac{\bar{F}^*_{n-3}}{F^*_0} & x & \vdots & \vdots & & & & \iddots & 1 \\ -\frac{\bar{F}^*_{n-4}}{F^*_0} & y & \vdots & \vdots & & & \iddots & 1 & 0 \\ 0 & 0 & \vdots & \vdots & & \iddots & 1 & 1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & 1 & -1 & \iddots & \vdots \\ \vdots & \vdots & 0 & 1 & 1 & -1 & \iddots & & \vdots \\ 0 & 0 & 1 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$$
\mathcal{N}=\begin{pmatrix}
1 & 0 & -\frac{K_2}{\bar{F}_0^*} & -\frac{F_{n-4}^*}{\bar{F}_0^*} & -\frac{F_{n-5}^*}{\bar{F}_0^*} & \cdots & -\frac{F_2^*}{\bar{F}_0^*} & -\frac{F_1^*}{\bar{F}_0^*} & -\frac{F_0^*}{\bar{F}_0^*} \\
0 & 1 & -\frac{K_1}{F_0^*} & -\frac{F_{n-3}^*}{F_0^*} & -\frac{F_{n-4}^*}{F_0^*} & \cdots & -\frac{F_3^*}{F_0^*} & -\frac{F_2^*}{F_0^*} & -\frac{F_1^*}{F_0^*} \\
\vdots & 0 & \Delta_{n-3} & 0 & \cdots & \cdots & \cdots & 0 & 1 \\
\vdots & \vdots & \Delta_{n-4} & \vdots & & & \iddots & 1 & 0 \\
\vdots & \vdots & \Delta_{n-5} & \vdots & & \iddots & 1 & \iddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & & \vdots \\
\vdots & \vdots & \Delta_2 & 0 & 1 & \iddots & & & \vdots \\
\vdots & \vdots & \Delta_1 & 1 & \iddots & & & & \vdots \\
0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}.
$$

We can observe that the inverse matrix of $\mathbf{\Phi}$ is $\mathbf{\Phi}^{-1}=\begin{pmatrix} 0 & \frac{1}{\bar{F}_0^*} \\ \frac{1}{F_0^*} & 0 \end{pmatrix}$.

Let $\mathbf{\Lambda}$ be partitioned as

$$
\mathbf{\Lambda}=\left(\begin{array}{c:c} \mathcal{J} & V \\ \hdashline U & \mathcal{B} \end{array}\right)
$$

$$
=\left(\begin{array}{cc:ccccccc}
c & \beta_{n-3} & \beta_{n-4} & \beta_{n-5} & \cdots & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\
d & \omega_{n-3} & \omega_{n-4} & \omega_{n-5} & \cdots & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\
\hdashline
0 & \alpha & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & t_0 & \alpha & -1 & \ddots & & & & \vdots \\
\vdots & t_1 & t_0 & \alpha & \ddots & \ddots & & & \vdots \\
\vdots & \vdots & t_1 & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \alpha & -1 & \ddots & \vdots \\
\vdots & \vdots & \vdots & & \ddots & t_0 & \alpha & -1 & 0 \\
0 & t_{n-6} & t_{n-7} & \cdots & \cdots & t_1 & t_0 & \alpha & -1
\end{array}\right)
\begin{matrix} \\ \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\\1\end{matrix}}\right\} n-4 \end{matrix}
\quad \text{(last } n-4 \text{ columns: } \underbrace{\qquad}_{n-4}\text{)} .
$$

According to Lemma 1.1 in [17], we have

$$
\mathcal{B}^{-1}=\begin{pmatrix}
b_1 & 0 & \cdots & \cdots & \cdots & 0 \\
b_2 & b_1 & \ddots & & & \vdots \\
b_3 & b_2 & b_1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
b_{n-4} & \cdots & \cdots & b_3 & b_2 & b_1
\end{pmatrix},
$$

where $b_i=-\det p_{i-1}(1\leqslant i\leqslant n-4)$, $\det p_i(0\leqslant i\leqslant n-5)$ is the determinant of Hessenberg matrix, and can be calculated by Lemma 1.1.

Suppose that $\mathbf{T}_{F_n^*}$ is invertible, and so is $\mathbf{\Lambda}$. Since $\mathcal{B}$ is invertible, the Schur complement of $\mathcal{B}$, denoted by $\mathcal{S}$, is also invertible, and we have

$$\begin{aligned}
\mathcal{S} &= \mathcal{J} - V\mathcal{B}^{-1}U \\
&= \begin{pmatrix} c & \beta_{n-3} \\ d & \omega_{n-3} \end{pmatrix} - \begin{pmatrix} \beta_{n-4} & \beta_{n-5} & \cdots & \beta_1 \\ \omega_{n-4} & \omega_{n-5} & \cdots & \omega_1 \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & \cdots & 0 \\ b_2 & b_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ b_{n-4} & \cdots & \cdots & b_2 & b_1 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 0 & t_0 \\ \vdots & t_1 \\ \vdots & t_2 \\ \vdots & \vdots \\ 0 & t_{n-6} \end{pmatrix} \\
&= \begin{pmatrix} c & \beta_{n-3} - \beta_{n-4}b_1\alpha - \sum_{k=1}^{n-5} \beta_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j}) \\ d & \omega_{n-3} - \omega_{n-4}b_1\alpha - \sum_{k=1}^{n-5} \omega_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j}) \end{pmatrix},
\end{aligned}$$

with the determinant:

$$\begin{aligned}
s = \det \mathcal{S} = c&\left(\omega_{n-3} - \omega_{n-4}b_1\alpha - \sum_{k=1}^{n-5} \omega_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right) \\
&- d\left(\beta_{n-3} - \beta_{n-4}b_1\alpha - \sum_{k=1}^{n-5} \beta_k(b_{n-3-k}\alpha + \sum_{j=1}^{n-4-k} b_j t_{n-4-k-j})\right).
\end{aligned}$$

Hence, the inverse of $\mathbf{\Lambda}$ is given with the following form:

$$\mathbf{\Lambda}^{-1} = [a_{i,j}]_{1\leqslant i,j\leqslant n-2} = \begin{pmatrix} \mathcal{S}^{-1} & -\mathcal{S}^{-1}V\mathcal{B}^{-1} \\ -\mathcal{B}^{-1}U\mathcal{S}^{-1} & \mathcal{B}^{-1}U\mathcal{S}^{-1}V\mathcal{B}^{-1} + \mathcal{B}^{-1} \end{pmatrix},$$

where $[a_{i,j}]_{1\leqslant i,j\leqslant n-2}$ are the same as in Theorem 2.2.

We obtain that

$$\mathbf{\Phi}^{-1} \oplus \mathbf{\Lambda}^{-1} = \begin{pmatrix} 0 & \frac{1}{\bar{F}_0^*} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{1}{F_0^*} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-3} & a_{1,n-2} \\ \vdots & \vdots & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-3} & a_{2,n-2} \\ \vdots & \vdots & a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-3} & a_{3,n-2} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & a_{n-3,1} & a_{n-3,2} & a_{n-3,3} & \cdots & a_{n-3,n-3} & a_{n-3,n-2} \\ 0 & 0 & a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-3} & a_{n-2,n-2} \end{pmatrix},$$

and we have

$$\mathbf{T}_{F_n^*}^{-1} = \mathcal{N}(\mathbf{\Phi}^{-1} \oplus \mathbf{\Lambda}^{-1})\mathcal{M} = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} & \cdots & \psi_{1,n-2} & \psi_{1,n-1} & \psi_{1,n} \\ \bar{\psi}_{1,2} & \psi_{2,2} & \psi_{2,3} & \cdots & \psi_{2,n-2} & \psi_{2,n-1} & \psi_{1,n-1} \\ \bar{\psi}_{1,3} & \bar{\psi}_{2,3} & \psi_{3,3} & \cdots & \psi_{3,n-3} & \psi_{2,n-2} & \psi_{1,n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \bar{\psi}_{1,n-2} & \bar{\psi}_{2,n-2} & \bar{\psi}_{3,n-3} & \cdots & \psi_{3,3} & \psi_{2,3} & \psi_{1,3} \\ \bar{\psi}_{1,n-1} & \bar{\psi}_{2,n-1} & \bar{\psi}_{2,n-2} & \cdots & \bar{\psi}_{2,3} & \psi_{2,2} & \psi_{1,2} \\ \bar{\psi}_{1,n} & \bar{\psi}_{1,n-1} & \bar{\psi}_{1,n-2} & \cdots & \bar{\psi}_{1,3} & \bar{\psi}_{1,2} & \psi_{1,1} \end{pmatrix},$$

where $[\psi_{i,j}](1 \leqslant i \leqslant [\frac{n+1}{2}];\ i \leqslant j \leqslant n+1-j)$ are the same as in Theorem 2.2.

Which completes the proof. □

3 Numerical Example

In this section, an example demonstrates the method which introduced above for the calculation of determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix. Here we consider a 7×7 matrix:

$$\mathbf{T}_{F_7^*} = \begin{pmatrix} 0 & i & 1+i & 1+2i & 2+3i & 3+5i & 5+8i \\ -i & 0 & i & 1+i & 1+2i & 2+3i & 3+5i \\ 1-i & -i & 0 & i & 1+i & 1+2i & 2+3i \\ 1-2i & 1-i & -i & 0 & i & 1+i & 1+2i \\ 2-3i & 1-2i & 1-i & -i & 0 & i & 1+i \\ 3-5i & 2-3i & 1-2i & 1-i & -i & 0 & i \\ 5-8i & 3-5i & 2-3i & 1-2i & 1-i & -i & 0 \end{pmatrix}_{7\times 7}.$$

Using the corresponding formulas in Theorem 2.1, we get

$$\begin{aligned} &x = -8-5i,\ y = -5-3i,\ \Delta_1 = 1+2i,\ \alpha = 2i,\ t_0 = 1+2i,\ t_1 = 2+2i, \\ &K_1 = -30-21i,\ K_3 = -20+172i,\ K_4 = -11+108i, \end{aligned}$$

and from Lemma 1.1, we can obtain

$$\begin{aligned} \det \nabla_4([\eta_k]_{k=1}^4, -1, \alpha, t_0, t_1) &= -10-40i, \\ \det \nabla_4([\xi_k]_{k=1}^4, -1, \alpha, t_0, t_1) &= -15-64i, \\ \det \nabla_4([F_k^*]_{k=1}^4, -1, \alpha, t_0, t_1) &= 14+3i. \end{aligned}$$

From (2.1), we obtain

$$\begin{aligned} \det \mathbf{T}_{F_7^*} &= -\bar{F}_0^*[F_0^*(K_3 \det \nabla_4([\eta_k]_{k=1}^4, -1, \alpha, t_0, t_1) - K_4 \det \nabla_4([\xi_k]_{k=1}^4, -1, \alpha, t_0, t_1)) \\ &\quad - \bar{F}_4^*(K_1 \det \nabla_4([\eta_k]_{k=1}^4, -1, \alpha, t_0, t_1) - K_4 \det \nabla_4([F_k^*]_{k=1}^4, -1, \alpha, t_0, t_1)) \\ &\quad + \bar{F}_3^*(K_1 \det \nabla_4([\xi_k]_{k=1}^4, -1, \alpha, t_0, t_1) - K_3 \det \nabla_4([F_k^*]_{k=1}^4, -1, \alpha, t_0, t_1))] \\ &= 32. \end{aligned}$$

As the inverse calculation, if we use the corresponding formulas in Theorem 2.2, we get

$$\psi_{1,1}=-\tfrac{13}{32},\ \psi_{1,2}=\tfrac{1}{8}-\tfrac{3}{32}i,\ \psi_{1,3}=\tfrac{25}{32}-\tfrac{9}{16}i,\ \psi_{1,4}=\tfrac{5}{8}+\tfrac{29}{32}i,$$

$$\psi_{1,5}=-\tfrac{9}{32}+\tfrac{11}{16}i,\ \psi_{1,6}=-\tfrac{7}{8}-\tfrac{3}{32}i,\ \psi_{1,7}=\tfrac{13}{32}-\tfrac{1}{4}i,\ \psi_{2,2}=\tfrac{3}{32},$$

$$\psi_{2,3}=-\tfrac{17}{16}+\tfrac{17}{32}i,\ \psi_{2,4}=\tfrac{3}{32}-\tfrac{3}{2}i,\ \psi_{2,5}=\tfrac{15}{16}-\tfrac{17}{32}i,\ \psi_{2,6}=\tfrac{35}{32}+i,$$

$$\psi_{3,3}=-\tfrac{9}{32},\ \psi_{3,4}=-\tfrac{9}{16}-\tfrac{17}{32}i,\ \psi_{3,5}=\tfrac{1}{32}-\tfrac{5}{8}i,\ \psi_{4,4}=-\tfrac{61}{32},$$

so we can get

$$\mathbf{T}_{F_7^*}^{-1}=\begin{pmatrix} -\frac{13}{32} & \frac{1}{8}-\frac{3}{32}i & \frac{25}{32}-\frac{9}{16}i & \frac{5}{8}+\frac{29}{32}i & -\frac{9}{32}+\frac{11}{16}i & -\frac{7}{8}-\frac{3}{32}i & \frac{13}{32}-\frac{1}{4}i \\ \frac{1}{8}+\frac{3}{32}i & \frac{3}{32} & -\frac{17}{16}+\frac{17}{32}i & \frac{3}{32}-\frac{3}{2}i & \frac{15}{16}-\frac{17}{32}i & \frac{35}{32}+i & -\frac{7}{8}-\frac{3}{32}i \\ \frac{25}{32}+\frac{9}{16}i & -\frac{17}{16}-\frac{17}{32}i & -\frac{9}{32} & -\frac{9}{16}-\frac{17}{32}i & \frac{1}{32}-\frac{5}{8}i & \frac{15}{16}-\frac{17}{32}i & -\frac{9}{32}+\frac{11}{16}i \\ \frac{5}{8}-\frac{29}{32}i & \frac{3}{32}+\frac{3}{2}i & -\frac{9}{16}+\frac{17}{32}i & -\frac{61}{32} & -\frac{9}{16}-\frac{17}{32}i & \frac{3}{32}-\frac{3}{2}i & \frac{5}{8}+\frac{29}{32}i \\ -\frac{9}{32}-\frac{11}{16}i & \frac{15}{16}+\frac{17}{32}i & \frac{1}{32}+\frac{5}{8}i & -\frac{9}{16}+\frac{17}{32}i & -\frac{9}{32} & -\frac{17}{16}+\frac{17}{32}i & \frac{25}{32}-\frac{9}{16}i \\ -\frac{7}{8}+\frac{3}{32}i & \frac{35}{32}-i & \frac{15}{16}+\frac{17}{32}i & \frac{3}{32}+\frac{3}{2}i & -\frac{17}{16}-\frac{17}{32}i & \frac{3}{32} & \frac{1}{8}-\frac{3}{32}i \\ \frac{13}{32}+\frac{1}{4}i & -\frac{7}{8}+\frac{3}{32}i & -\frac{9}{32}-\frac{11}{16}i & \frac{5}{8}-\frac{29}{32}i & \frac{25}{32}+\frac{9}{16}i & \frac{1}{8}+\frac{3}{32}i & -\frac{13}{32} \end{pmatrix}_{7\times 7}.$$

4 Conclusion

In this paper, by constructing the special transformation matrices we give the determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix in section 2.

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Competing Interests

Authors have declared that no competing interests exist.

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