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Research Article

Some Convergence Results for a Class of Generalized Nonexpansive Mappings in Banach Spaces

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This paper investigates fixed points of Reich-Suzuki-type nonexpansive mappings in the context of uniformly convex Banach spaces through an M^* iterative method. Under some appropriate situations, some strong and weak convergence theorems are established. To support our results, a new example of Reich-Suzuki-type nonexpansive mappings is presented which exceeds the class of Suzuki-type nonexpansive mappings. The presented results extend some recently announced results of current literature.

1. Introduction

Fixed point theory and applications played an important role in many areas of applied sciences and solved many problems rising in engineering, mathematical economics and optimization. When the problem cannot be solved by ordinary analytical methods, we transform it to the form of fixed point problems and apply an appropriate iterative method to obtain the required fixed point. In 1922, Banach [1] proved that if T is contraction mapping on a closed subset K of a Banach space, that is, $\|Ts - Ts'\| \leq \alpha\|s - s'\|$ for a fixed $\alpha \in [0, 1)$ and $s, s' \in K$, then T possesses a unique fixed point, which can be obtained by using the Picard [2] iteration process. In 1930, Caccioppoli [3] extended the Banach result to the frame work of complete metric spaces. The Banach-Caccioppoli results is an important tool for solving many problems in fractional calculus, mathematical biology, mathematical economics, and engineering. Nevertheless, the conclusions of Banach-Caccioppoli result no more holds if one replace the contraction condition of T by a nonexpansive condition of T, that is, if one choose the value of $\alpha = 1$ in the contraction inequality, even the underlying space is a Banach space. The first basic result concerning the existence of a fixed point for the class of nonexpansive mappings was independently proved by Browder [4] and Göhde [5]. They proved that any nonexpansive mapping on a closed bounded convex subset of a uniformly convex Banach space (in short UCBS) always possesses a fixed point. The class of nonexpansive mappings is an important extension of the class of contraction mappings. On the other hand, Suzuki [6] introduced the notion of generalized nonexpansive mappings by restricting the range of elements that satisfy the nonexpansive inequality: a selfmap T on a subset K of a Banach space is said to be Suzuki-type nonexpansive if for all $s, s' \in K$ satisfying the condition $1/2\|s - Ts\| \leq \|s - s'\|$, one has the inequality $\|Ts - Ts'\| \leq \|s - s'\|$. Obviously, any nonexpansive mapping T belongs to the class of Suzuki-type nonexpansive mappings. Recently in 2019, Pant and Pandey [7] introduced an interesting generalization of Suzuki-type nonexpansie mappings: a selfmap T on a subset K of a Banach space is said to be Reich-Suzuki-type nonexpansive if for all $s, s' \in K$, satisfying the condition $1/2\|s - Ts\| \le \|s - s'\|$, one can find a real constant $c \in [0, 1)$, such that $\|Ts - Ts'\| \le c\|s - Ts\| + \|s' - Ts'\| + (1 - 2c)\|s - s'\|$. The class of Reich-Suzuki-type nonexpansive mappings is important, because it properly contains the class of Suzuki-type nonexpansive mappings as shown by an example in this paper. Unlike contraction mappings, Picard iterative process does not always converge to a fixed point of a nonexpansive mapping T even $F(T) = \{p \in K : Tp = p\} \ne \emptyset$. For finding fixed points of nonexpansive and generalized nonexpansive mappings, many iterative processes are available in the literature (see, e.g., Mann [8], Ishikawa [9], Agarwal et al. [10], Noor [11], Abbas and Nazir [12], and Thakur et al. [13]). For more details on this direction, we shall refer the reader to [14–18].

Recently, Ullah and Arshad [19] introduced a new iterative process called M^* iterative process, as follows:

$$\left.\begin{aligned} & s_1 \in K, \\ & z_n = (1 - \rho_n)s_n + \rho_n T s_n, \\ & y_n = T((1 - \delta_n)s_n + \delta_n T z_n), \\ & s_{n+1} = T y_n). \end{aligned}\right\} \quad (1)$$

They proved that M^* iterative process can be used for finding fixed points of Suzuki-type nonexpansive mappings. With the help of a numerical example, they proved that M^* iterative process converges faster that all of the Mann, Ishikawa, Agarwal, Noor, Abbas, and Thakur iterative processes. In this paper, we extend their results to the general setting of Reich-Suzuki-type nonexpansive mappings.

2. Preliminaries

This section is consists of some basic definitions and earlier results, which are needed in sequel.

Definition 1. Let $\{s_n\}$ be a bounded sequence in a Banach space X and $\emptyset \ne K \subseteq X$. The asymptotic radius of $\{s_n\}$ wrt to K is $r(K, \{s_n\}) = \inf\{\limsup_{n\longrightarrow\infty}\|s_n - w\|: w \in K\}$. Moreover, the asymptotic center of $\{s_n\}$ relative to K is the set $A(K, s_n\}) = \{w \in K : \limsup_{n\longrightarrow\infty}\|s_n - w\| = r(K, s_n)\}$.

Remark 2. In Banach spaces, the set $A(K, \{s_n\})$ is singleton provided that X is uniformly convex [20]. Moreover, the set $A(K, \{s_n\})$ is convex provided that K is weakly compact and convex (see, e.g., [21, 22]).

Definition 3 (see [23]). A Banach space X is said to be endowed with an Opial's property provided that for any sequence $\{s_n\}$ in X which weakly converges to $s \in X$ and for each element w of X - $\{s\}$, it follows that

$$\limsup_{n\longrightarrow\infty}\|s_n - s\| < \limsup_{n\longrightarrow\infty}\|s_n - w\|. \quad (2)$$

The following result gives many numbers of Reich-Suzuki-type nonexpansive mappings.

Lemma 4 (see [7]). *Let K be a nonempty subset of a Banach space and let $T : K \longrightarrow K$ be a Reich-Suzuki-type nonexpansive mapping. Then, T is also Suzuki-type nonexpansive mapping with a real constant $c = 0$.*

Lemma 5 (see [7]). *Let K be a nonempty subset of a Banach space and let $T : K \longrightarrow K$ be a Reich-Suzuki-type nonexpansive mapping. Then, for all $p \in F(T)$ and $s \in K$, we have $\|Ts - p\| \le \|s - p\|$.*

Lemma 6 (see [7]. *Let K be a nonempty subset of a Banach space and let $T : K \longrightarrow K$ be a Reich-Suzuki-type nonexpansive mapping. Then, for all $s, s' \in K$, we have $\|s - Ts'\| \le (3 + c/1 - c)\|s - Ts\| + \|s - s'\|$.*

Lemma 7 (see [15]). *Let T be a Reich-Suzuki-type nonexpansive mapping on a subset K of a Banach space X with the Opial property. If $\{s_n\}$ converges weakly to q and $\lim_{n\longrightarrow\infty}\|s_n - Ts_n\| = 0$, then $q \in F(T)$.*

We recall an important result of Schu [24].

Lemma 8. *Let X be a UCBS and $0 < a \le d_n \le b < 1$ for all $n \in \mathbb{N}$. If $\{g_n\}$ and $\{h_n\}$ are two sequences in X such that $\limsup_{n\longrightarrow\infty}\|g_n\| \le \xi$, $\limsup_{n\longrightarrow\infty}\|h_n\| \le \xi$, and $\lim_{n\longrightarrow\infty}\|d_n g_n + (1 - d_n)h_n\| = \xi$ for some $\xi \ge 0$, then $\lim_{n\longrightarrow\infty}\|g_n - h_n\| = 0$.*

3. Convergence Theorems in Uniformly Convex Banach Spaces

This section deals with some weak and strong convergence results. First, we give the following key lemma, which will play an important role in the sequel.

Lemma 9. *Let K be a nonempty closed convex subset of a Banach space X and let $T : K \longrightarrow K$ be a Reich-Suzuki-type nonexpansive mapping with $F(T) \ne \emptyset$. For arbitrarily chosen $s_1 \in K$, let the sequence $\{s_n\}$ be defined by (1); then, $\lim_{n\longrightarrow\infty}\|s_n - q\|$ exists for all $q \in F(T)$.*

Proof. Suppose $q \in F(T)$. By Lemma 5, we have

$$\begin{aligned} \|s_{n+1} - q\| &= \|Ty_n - q\| \le \|y_n - q\| = \|T((1 - \delta_n)s_n + \delta_n T z_n) - q\| \\ &\le \|(1 - \delta_n)s_n + \delta_n T z_n - q\| \le (1 - \delta_n)\|s_n - q\| + \delta_n\|z_n - q\| \\ &= (1 - \delta_n)\|s_n - q\| + \delta_n(\|(1 - \rho_n)s_n + \rho_n T s_n - q\|) \\ &\le (1 - \delta_n)\|s_n - q\| + \delta_n((1 - \rho_n)\|s_n - q\| + \rho_n\|Ts_n - q\|) \\ &\le (1 - \delta_n)\|s_n - q\| + \delta_n((1 - \rho_n)\|s_n - q\| + \rho_n\|s_n - q\|) \\ &= (1 - \delta_n)\|s_n - q\| + \delta_n(\|s_n - q\|) = \|s_n - q\|. \end{aligned} \quad (3)$$

Thus, $\{\|s_n - q\|\}$ is bounded and nonincreasing, which implies that $\lim_{n\longrightarrow\infty}\|s_n - q\|$ exists for each $q \in F(T)$.

Now, we establish the following theorem, which will be used in the upcoming strong and weak convergence results.

Theorem 10. *Let K be a nonempty closed convex subset of a UCBS X and let $T : K \longrightarrow K$ be a Reich-Suzuki-type nonexpansive mapping. Let $\{s_n\}$ be the sequence defined by (1). Then, $F(T) \neq \emptyset$ if and only if $\{s_n\}$ is bounded and $\lim_{n\longrightarrow\infty} \|s_n - Ts_n\| = 0$.*

Proof. Assume that the sequence $\{s_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|s_n - Ts_n\| = 0$. Fix $q \in A(K, \{s_n\})$. We need to prove that q is the element of $F(T)$. Using Lemma 6, we have

$$r(Tq, \{s_n\}) = \limsup_{n\longrightarrow\infty}\|s_n - Tq\| \limsup_{n\longrightarrow\infty}\left(\frac{3+c}{1-c}\right)\|s_n - Ts_n\| + \limsup_{n\longrightarrow\infty}\|s_n - q\| = \limsup_{n\longrightarrow\infty}\|s_n - q\| = r(q, \{s_n\}). \tag{4}$$

Hence, $Tq \in A(K, \{s_n\}$. Since the set $A(K, \{s_n\})$ is singleton, we must have $Tq = q$. Hence, p is the element of $F(T)$ and so $F(T) \neq \emptyset$.

Conversely, we take the set $F(T) \neq \emptyset$ and prove that the sequence $\{s_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|s_n - Ts_n\| = 0$. Fix $q \in F(T)$. By Lemma 9, $\lim_{n\longrightarrow\infty}\|s_n - q\|$ exists and $\{s_n\}$ is bounded. Let us put

$$\lim_{n\longrightarrow\infty}\|s_n - q\| = \xi. \tag{5}$$

In the proof of Lemma 9, we see that

$$\|z_n - q\| \leq \|s_n - q\| \Rightarrow \limsup_{n\longrightarrow\infty}\|z_n - q\| \leq \limsup_{n\longrightarrow\infty}\|s_n - q\| = \xi. \tag{6}$$

Using Lemma 5 and (5), we obtain the following:

$$\limsup_{n\longrightarrow\infty}\|Ts_n - q\| \leq \limsup_{n\longrightarrow\infty}\|s_n - q\| = \xi. \tag{7}$$

Again in the from the proof of Lemma 9, we see that

$$\|s_{n+1} - q\| \leq (1-\delta_n)\|s_n - q\| + \delta_n\|z_n - q\|. \tag{8}$$

It follows that

$$\|s_{n+1} - q\| - \|s_n - q\| \leq \frac{\|s_{n+1} - q\| - \|s_n - q\|}{\delta_n} \leq \|z_n - q\| - \|s_n - q\|. \tag{9}$$

So, we can get $\|s_{n+1} - q\| \leq \|z_n - q\|$.

$$\Rightarrow \xi \leq \liminf_{n\longrightarrow\infty}\|z_n - q\|. \tag{10}$$

From (6) and (10), we get

$$\xi = \lim_{n\longrightarrow\infty}\|z_n - q\|. \tag{11}$$

Using (11), we have

$$\begin{aligned}\xi &= \lim_{n\longrightarrow\infty}\|z_n - q\| = \lim_{n\longrightarrow\infty}\|(1-\rho_n)s_n + \rho_n Ts_n - q\| \\ &= \lim_{n\longrightarrow\infty}\|(1-\rho_n)(s_n - q) + \rho_n(Ts_n - q)\|.\end{aligned} \tag{12}$$

Hence

$$\xi = \lim_{n\longrightarrow\infty}\|(1-\rho_n)(s_n - q) + \rho_n(Ts_n - q)\|. \tag{13}$$

Now bearing (5), (7), and (13) and Lemma 8, we obtain

$$\lim_{n\longrightarrow\infty}\|Ts_n - s_n\| = 0. \tag{14}$$

Now, we state and prove the following strong convergence results, under the strong assumption of compactness. Notice that it is the extension of the ([19], Theorem 3.3) from the setting of Suzuki-type nonexpansive mappings to the general setting of Reich-Suzuki-type nonexpansive mappings.

Theorem 11. *Let K be a nonempty convex compact subset of a UCBS X and let T and $\{s_n\}$ be as in Theorem 10 and $F(T) \neq \emptyset$. Then, $\{s_n\}$ converges strongly to the fixed point of T.*

Proof. By Theorem 10, $\lim_{n\longrightarrow\infty}\|Ts_n - s_n\| = 0$. The compactness of K follows that the sequence $\{s_n\}$ has a strongly convergent subsequence, namely, $\{s_{n_l}\}$ with a strong limit, say, $v \in K$. We need to prove that v is the strong limit of $\{s_n\}$ and $v \in F(T)$. Using Lemma 6, we have

$$\|s_{n_l} - Tv\| \leq \left(\frac{3+c}{1-c}\right)\|s_{n_l} - Ts_{n_l}\| + \|s_{n_l} - v\|. \tag{15}$$

If we apply $l \longrightarrow \infty$, then we obtain $Tv = v$, that is, $v \in F(T)$. By Lemma 9, $\lim_{n\longrightarrow\infty}\|s_n - v\|$ exists. Hence, v is the strong limit of $\{s_n\}$.

Now, we shall only state the following result. We will not include the proof here because the proof is elementary.

Theorem 12. *Let K be a nonempty closed convex subset of a UCBS X and let T and $\{s_n\}$ be as in Theorem 10. If $F(T) \neq \emptyset$ and $\liminf_{n\longrightarrow\infty}(\inf_{p\in F(T)}\|s_n - p\|) = 0$, then $\{s_n\}$ converges strongly to fixed point of T.*

The following definition is essentially due to Sentor and Dotson [25].

Definition 13. A mapping T on a subset K of a Banach space X is said to satisfy condition (I) if there exists some nondecreasing function $\eta : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\eta(0) = 0$, $\eta(a) > 0$ for every $a > 0$ and $\|s - Ts\| \geq \eta(\inf_{q\in F(T)}\|s - q\|)$ for all $s \in K$.

The strong convergence result using condition (I) is given as follows. Notice that it is the extension of the [19], Theorem 3.4, from the setting of Suzuki-type nonexpansive mappings to the general setting of Reich-Suzuki-type nonexpansive mappings.

Theorem 14. *Let K be a nonempty closed convex subset of UCBS X and let T and $\{s_n\}$ be as in Theorem 10 and $F(T) \neq \varnothing$. If T satisfies condition (I), then $\{s_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 10, it follows that

$$\liminf_{n\longrightarrow\infty} \| Ts_n - s_n \| = 0. \tag{16}$$

Since T is mapping satisfying condition (I), so we have

$$\|s_n - Ts_n\| \geq \eta\left(\left(\inf_{q\in F(T)} \|s_n - q\|\right)\right). \tag{17}$$

Using (16) and (17), we have

$$\liminf_{n\longrightarrow\infty} \eta\left(\left(\inf_{q\in F(T)} \|s_n - q\|\right)\right) = 0. \tag{18}$$

But the function η is nondecreasing satisfying $\eta(0) = 0$, so we have

$$\liminf_{n\longrightarrow\infty} \left(\inf_{q\in F(T)} \|s_n - q\|\right) = 0. \tag{19}$$

The conclusions follows from Theorem 12.

We finish this section with following weak convergence result. Notice that it is the extension of the [19], Theorem 3.2, from the setting of Suzuki-type nonexpansive mappings to the general setting of Reich-Suzuki-type nonexpansive mappings.

Theorem 15. *Let X be a UCBS with Opial's property, K a nonempty closed convex subset of X, and T and $\{s_n\}$ as in Theorem 10 and $F(T) \neq \varnothing$. Then $\{s_n\}$ converges weakly to a fixed point of T.*

Proof. By Theorem 10, $\{s_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|Ts_n - s_n\| = 0$. The uniform convexity of X follows the reflexivity of X. Hence, the sequence $\{s_n\}$ must have a weakly convergenet subsequence, namely, $\{s_{n_k}\}$ with a weak limit say, $u_1 \in K$. By Lemma 7, u_1 is the element of the set $F(T)$. We need only to show that u_1 is the weak limit of the sequence $\{s_n\}$ itself. If u_1 is not the weak limit of $\{s_n\}$, then one can choose a subsequence, namely, $\{s_{n_l}\}$ of $\{s_n\}$ and $u_2 \in K$ such that $\{s_{n_l}\}$ converges weakly to u_2 and $u_2 \neq u_1$. Again by Lemma 7, u_2 is the element of $F(T)$. By Lemma 9 and the Opial property, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|s_n - u_1\| &= \lim_{k\longrightarrow\infty}\left\|s_{n_k} - u_1\right\| < \lim_{k\longrightarrow\infty}\left\|s_{n_k} - u_2\right\| \\ &= \lim_{n\longrightarrow\infty}\|s_n - u_2\| = \lim_{l\longrightarrow\infty}\left\|s_{n_l} - u_2\right\| \\ &< \lim_{l\longrightarrow\infty}\left\|s_{n_l} - u_1\right\| = \lim_{n\longrightarrow\infty}\|s_n - u_1\|.\end{aligned} \tag{20}$$

This is a contradiction. Therefore, the proof is complete.

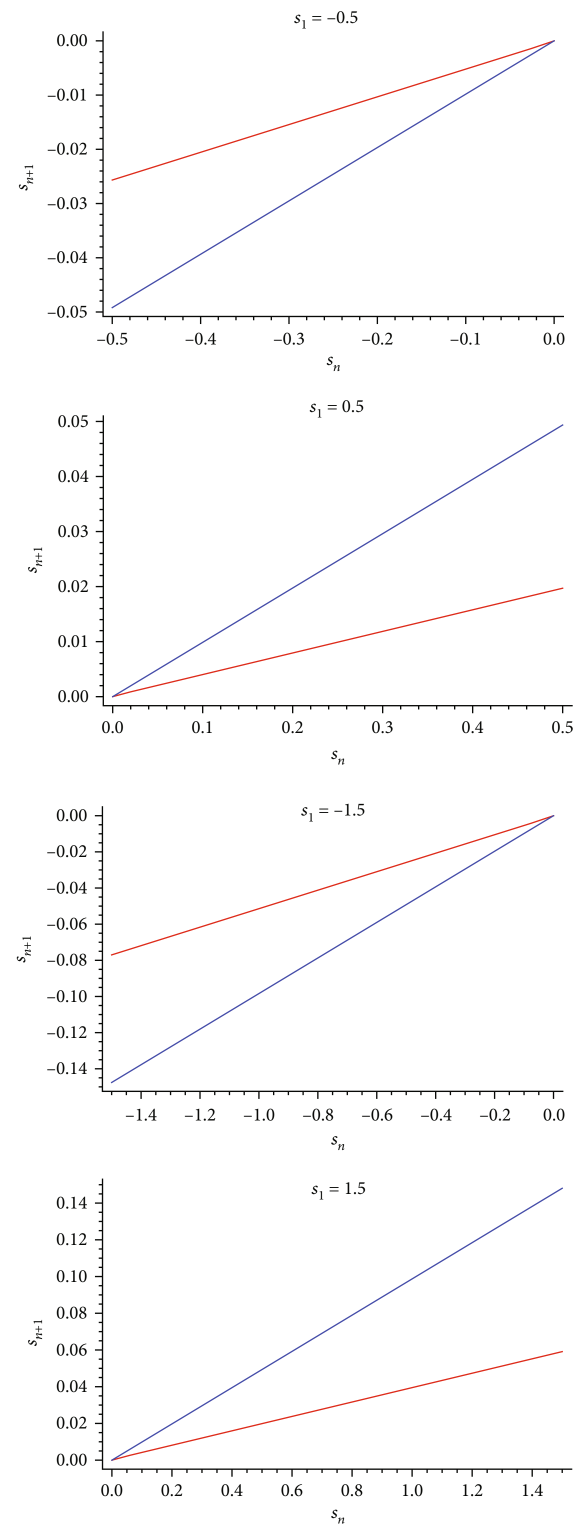

Figure 1: Rate of convergence of M^* (red line) and Thakur (blue line) under different initial points.

4. Numerical Example

In this section, first we present a new example of Reich-Suzuki-type nonexpansive mappings, which exceeds the class of Suzuki-type nonexpansive mappings as follows.

Table 1: Influence of parameters: comparison of various iteration processes.

Iterations	Initial points					
	−1.9	−1.6	−1.1	1.2	1.6	1.9
For $\delta_n = n/(n+1)^{10/9}$, $\rho_n = 1/(n+3)^{2/3}$						
S	16	16	16	15	16	16
Thakur	9	9	8	9	9	9
M^*	**7**	**7**	**6**	**6**	**6**	**6**
For $\delta_n = 1 - 1/\sqrt{5n+3}$, $\rho_n = 1/n^3$						
S	18	18	18	15	15	15
Thakur	8	8	8	9	9	9
M^*	**5**	**5**	**5**	**5**	**5**	**5**
For $\delta_n = 1/n$, $\rho_n = 1/\sqrt{n+24}$						
S	18	18	1 8	17	18	18
Thakur	9	9	9	9	9	9
M^*	**8**	**8**	**8**	**8**	**8**	**8**
For $\delta_n = \sqrt{n/4n+3}$, $\rho_n = 1/(4n+9)^{3/4}$						
S	18	18	18	18	18	18
Thakur	10	9	9	9	10	10
M^*	**8**	**7**	**7**	**4**	**7**	**7**

Example 16. Define $T : [-2,2] \longrightarrow [-2,2]$ by

$$Ts = \begin{cases} -\dfrac{s}{2}, & \text{for } s \in [-2,0) \setminus \left\{-\dfrac{1}{4}\right\} \\ 0, & \text{for } s = -\dfrac{1}{4} \\ -\dfrac{s}{5} & \text{for } s \in [0,2]. \end{cases} \quad (21)$$

If we choose $s = -1/4$ and $s' = -2/5$, then $1/2|s - Ts| < |s - s'|$ and $|Ts - Ts'| > |s - s'|$. On the other hand, T is Reich-Suzuki-type nonexpansive with the constant $c = 1/2$. We shall include only nontrivial cases here.

Case 1. For $s, s' \in [-2,0) \setminus \{-1/4\}$, we have

$$\begin{aligned} \|Ts - Ts'\| &= \frac{1}{2}|s - s'| \le \frac{1}{2}|s| + \frac{1}{2}|s'| \le \frac{3}{4}|s| + \frac{3}{4}|s'| \\ &= \frac{1}{2}\left|s + \frac{s}{2}\right| + \frac{1}{2}\left|s' + \frac{s'}{2}\right| \\ &\le c\|s - Ts\| + c\|s' - Ts'\| + (1-2c)\|s - s'\|. \end{aligned} \quad (22)$$

Case 2. For $s, s' \in [0,2]$, we have

$$\begin{aligned} \|Ts - Ts'\| &= \frac{1}{5}|s - s'| \le \frac{1}{5}|s| + \frac{1}{5}|s'| \le \frac{6}{10}|s| + \frac{6}{10}|s'| \\ &= \frac{1}{2}\left|s + \frac{s'}{5}\right| + \frac{1}{2}\left|s' + \frac{s'}{5}\right| \\ &= c\|s - Ts\| + c\|s' - Ts'\| + (1-2c)\|s - s'\|. \end{aligned} \quad (23)$$

Case 3. For $s \in [-2,0) \setminus \{-1/4\}$ and $s' \in [0,2]$, we have

$$\begin{aligned} \|Ts - Ts'\| &= \left|\frac{s}{5} - \frac{s'}{2}\right| \le \frac{1}{2}|s| + \frac{1}{5}|s'| \le \frac{3}{4}|s| + \frac{6}{10}|s'| \\ &= \frac{1}{2}\left|s + \frac{s}{2}\right| + \frac{1}{2}\left|s' + \frac{s'}{5}\right| \\ &= c\|s - Ts\| + c\|s' - Ts'\| + (1-2c)\|s - s'\|. \end{aligned} \quad (24)$$

Case 4. For $s \in [-2,0) \setminus \{-1/4\}$ and $s' = -1/4$, we have

$$\begin{aligned} \|Ts - Ts'\| &= \frac{1}{2}|s| \le \frac{3}{4}|s| \le \frac{3}{4}|s| + \frac{1}{8} = \frac{1}{2}\|s - Ts|\| + \frac{1}{2}\|s' - Ts'\| \\ &= c\|s - Ts\| + c\|s' - Ts'\| + (1-2c)\|s - s'\|. \end{aligned} \quad (25)$$

Case 5. For $s \in [0,2]$ and $s' = -1/4$, we have

$$\begin{aligned} \|Ts - Ts'\| &= \frac{1}{5}|s| \le \frac{6}{10}|s| \le \frac{6}{10}|s| + \frac{1}{8} = \frac{1}{2}\|s - Ts\| + \frac{1}{2}\|s' - Ts'\| \\ &= c\|s - Ts\| + c\|s' - Ts'\| + (1-2c)\|s - s'\|. \end{aligned} \quad (26)$$

Thus, T is a Reich-Suzuki-type nonexpansive mapping with $F(T) \neq \varnothing$.

Now, using the above example, and $\delta_n = 1/\sqrt{n+5}$ and $\rho_n = \sqrt{n}/(n+7)^{7/4}$, we get $\|s_n - p\| < 10^{-15}$, our stopping criterion where $p = 0$ is a fixed point of T. The graphs in Figure 1 show that the sequence $\{s_n\}$ generated by M^* iteration process converges faster than the sequence $\{s_n\}$ generated by the leading Thakur [13] iteration process.

Finally, we compare the numbers of iteration required to obtain a fixed points of M^* iteration with leading Thakur and S iterations. Set $\|s_n - p\| < 10^{-10}$ as a stopping criterion where $p = 0$ is a fixed point of the mapping T.

Remark 17. Figure 1 and Table 1 suggest that M^* is better than the Thakur and S iterative processes.

Data Availability

No data were used to support this study.

Conflicts of Interest

We have no conflicts of interest.

Authors' Contributions

T. Abdeljawad, K. Ullah, J. Ahmad, M. de la Sen, and M.N. Khan contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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